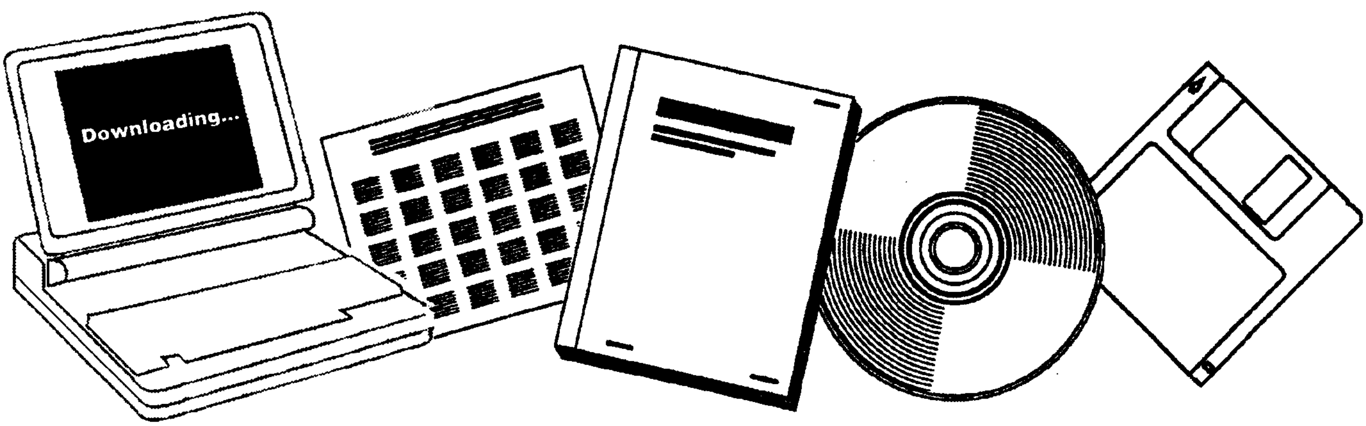
OPTIMIZATION IN THE STUDY OF A SINUSOIDAL WAVEFORM OF KNOWN PERIODICITY IN THE PRESENCE OF NOISE. APPLICATION IN RADIO ASTRONOMY, 1

SCIENTIFIC TRANSLATION SERVICE, SANTA
BARBARA, CALIF

OCT 1976



N7712955

NASA TECHNICAL TRANSLATION

NASA TT F-16764

OPTIMIZATION IN THE STUDY OF A SINUSOIDAL WAVEFORM OF KNOWN PERIODICITY IN THE PRESENCE OF NOISE (*). APPLICATION IN RADIO ASTRONOMY. I.

Marc Vinokur

Translation of "Optimisation dans la Recherche d'une Sinusoide de Periode Connue en Presence de Bruit (*). Application a la Radioastronomie. I.", Annales d' Astronomie, Vol. 28, No. 2, 1965, pp. 412-445.

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, DC 20546 October 1976

STANDARD TITLE PAGE

1. Report No. NASA TT F-16764	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle OPTIMIZATION IN THE STUDY OF A SINUSOIDAL WAVEFORM OF KNOWN PERIODICITY IN THE PRESENCE OF NOISE (*). APPLICATION IN RADIO ASTRONOMY I		5. Report Date October 1976
		6. Performing Organization Code
7. Author(s) Marc Vinokur, Paris-Meudon Observatory		8. Performing Organization Report No.
		10. Work Unit No.
9. Performing Organization Name and Address SCITRAN Box 5456 Santa Barbara, CA 93108		11. Contract or Grant No. NASw-2791
		13. Type of Report and Period Covered Translation
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		14. Sponsoring Agency Code

15. Supplementary Notes

Translation of "Optimisation dans la Recherche d'une Sinusoide de Periode Connue en Presence de Bruit (*). Application a la Radioastronomie. I, Annales d' Astronomie, Vol. 28, No. 2, 1965, pp. 412-445.

16. Abstract

In connection with the problems involved in the optimum utilization of information, the following deals with the measurement, in the presence of noise, of the amplitude and phase of a sinusoid whose period is known (with a DC signal as a limiting case). The aim is twofold: first, to achieve as good a precision as the signal to noise ratio and the observing time permit, and second, to attain such a precision with the minimum number of points (sampling) measured with the minimum accuracy (amplitude quantization). The following approaches are treated: least mean square curve fitting, optimization of the physical filter, convolution by a sinusoid having the same period as the signal, summation, FOURIER series analysis. It is shown that these methods, singly or suitably combined, lead to the same results, and the probability distribution functions of the amplitude and phase thus obtained are established. The effects of sampling and quantization on the final signal to noise ratio are then studied as a function of the characteristics of the filter preceding the recorder; several results are established or reproduced, and then applied to the case where filtering

17. Key Words (Selected by Author(s))		18. Distribution Statement Unclassified - Unlimited	
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages	22. Price

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is performed by integration over a finite interval of time.

OPTIMIZATION IN THE STUDY OF A SINUSOIDAL WAVEFORM OF KNOWN PERIODICITY IN THE PRESENCE OF NOISE (*). APPLICATION IN RADIO ASTRONOMY.

I. /4**

Marc Vinokur***

INTRODUCTION /4

Frequently the results of an observation during a physical phenomenon is a recording made of the sum:

- of a signal whose useful function is related to the characteristics of the phenomena and to those of the measuring instrument,

- and of the noise, parasitic random fluctuations whose statistical properties are invariant in time. Under these conditions, the problem is to arrive at the best possible understanding of the phenomenon that we wish to study. Generally, we have the following data at our disposal in order to do this: the recording itself with all the information it contains, the knowledge we have of the characteristics of the instrument, the evaluation that we have made, either theoretically or experimentally, of the statistical properties of the noise.

It could be that, in addition, we would have a certain amount of information on the particular peculiarities of the phenomenon or of

(*) Only the first part of the article is treated here. The application will be the topic of a future publication in this same journal.

** Numbers in margin indicate pagination in original foreign text.

*** Paris-Meudon Observatory

the signal.

The essential problem therefore consists of taking the best possible part of the set of these data and the means that we know of to exploit them. Two other questions are implicitly related to this problem of optimum use of the information:

- to evaluate the obtained degree of precision,
- to achieve such a degree of precision the most economically possible, that is, in particular, not to take but the minimum number of points to be recorded, and to measure these with a minimum of finesse.

Among the data that we have, a degree of importance is attached to those that we can obtain a priori about the particular peculiariti of the phenomenon or of the signal. It is on these that the choice o the methods and of the particular objectives to be achieved essentially depends. It is to their number and their quality that the precis that we hope to achieve is directly related.

Thus, for example, in the case that interests us, the fact that we know that the signal is sinusoidal already allows us to improve the signal-to-noise ratio by sampling the entire data with the use of the auto-correlation process [16 and 37]. When, in addition, we know in advance the period of this sinusoidal signal, then, and this is the topic of this article, it is possible to obtain an even great precision (often by several orders of magnitude).

Perhaps it is not useless to dwell on this question of the imp- ... of a priori data by briefly recalling their influence on the w certain problems relative to the interpretation of recordings of physical phenomena are approached.

First of all, let us consider a case where we know nothing of the phenomenon, of the signal, and even of the recording apparatus itself. Given the presence of noise, we cannot hope to do better them to discover some salient facts with a certain margin of error about their existence or at least a certain margin of error about the evaluation of the parameters on which they appear to depend. The processes of interpretation therefore boil down to a comparison of the recordings obtained from a certain number of models corresponding to simple hypotheses made about the nature of the phenomenon. This is the intuitive approach, that is, based on acquired experience of the known characteristics of the instrument and of the statistical properties of the noise.

It so happens that this interpretation is made difficult by the complexity of the phenomenon, the presence of data considered useless, or even the form in which the noise appears. We can therefore wish to modify the relative contribution of certain data relative to others, in order to be able to concentrate our attention on these which are of direct interest.

Among the methods generally used, we note smoothing, _a posteriori_ apodization, and deconvolution (or re-synthesis). All three of these techniques consist of a linear smoothing of the recorded frequency spectrum, a smoothing which evidently affects the characteristics of both the signal and the noise. The goal of smoothing is essentially to reduce the standard deviation of the noise which is achieved by reducing the width of frequency spectrum. The _a posteriori_ apodization has the same effect, but its purpose is totally different: it amounts, by reducing the relative importance of the high frequencies of the signal, to modifying the characteristics of the observing instrument (mirror in optics or antenna in radio astronomy), in the sense of a reduction of the secondary lobes with a corrollary effect of broadening the main lobe. Deconvolution, on the other hand, is generally aimed at improving the resolving power of the instrument by means of increasing the relative contribution of the high frequencies. We know that in the theoretical limit where there is no noise, this improvement can be theoretically as great as we wish, on the two conditions that the Fourier transform of the gain of the instrument does not vanish anywhere, and that it be perfectly continuous: under these conditions, in effect, there is no limitation on the quantity of data at our disposal. With this hypothesis, in effect, the problem is no longer to interpret a recording, but, on the contrary, to measure exactly the characteristics of the phenomenon. In the presence of noise, on the other hand, the quantity of data is necessarily limited, and this method is open to criticism because, on the one hand, it has characteristics in common with the other two cited above, of not adding anything more than is provided by the recording itself, except for a modification in the form in which the results are presented, and, on the other hand, it presents the risk of leading to interpretation errors, unless new experience had been acquired in the interpretation of recordings thus modified.

Altogether different is the research of ways to improve the results, such as the recording would provide by itself, by means of data external to this recording. These data could be related to the phenomenon itself. For example, in the case of radio astronomy, what we observe has the dimensions of energy, and the picture that we will propose of the phenomenon being studied therefore would not allow negative values. Used together with the data from the recording, such information should allow us to improve the resolving power of the instrument by a certain amount [4]. Similarly, we should take into account other possible data on the nature or the shape of the observed radio sources.

We are interested here in the case where the data at our dispos[al] concern the signal itself. For example, the simplest one is that of a DC signal: the application of the least squares criterion leads, therefore, to evaluating this quantity by integrating the recorded function, and the obtained precision is, as we know, proportional to the square root of the duration of observation. Based on this same criterion of minimum standard deviation, several cases have been studied, for example that where the signal itself is a random functi[on] [36 and 24] or the sum of a random function and of a polynomial of [kn]own degree [38, 18, 6], or the sum of gaussian functions [30 and ...] We could be led to using other approximation criteria [2, Chapter ...

in particular when we seek to improve the signal-to-noise ratio ... . We mention in this regard an original method used in seismography [23], which consists of using information that we have about the noise itself, more precisely, of its properties of statistical ...mory [5, p. 679].

The questions brought up in the first part of this article are linked in several respects to the above.

Relative first of all to the most general problem of processing the data from a given recording of signal and noise, it seems that the results obtained could prove to be useful. In effect, we know that any function can be identically represented in a given interval T, and only within this interval, by a Fourier series whose elements have periods that are sub-multiples of T (the DC term corresponds to the limiting case of a sinusoidal wave of frequency zero). Thus the signal itself is made up of the sum of an infinite number of sine waves whose periods are known in advance. The study of the recording of the signal, and of the noise of such sinusoidal waves and the evaluation of the degree of uncertainty due to the noise in the evaluation of their amplitudes and of their phases, is thus transposed from the domain of time to the domain of frequency for the set of data contained in the recording. Therefore it could eventually be more convenient to judge in this form the effects of the application of a given operator to this recording. This remark is especially applicable to the case of a periodic signal.

For the particular problem treated in this article, we have been led to establishing totally general relationships concerning the statistical properties of any results y(t) of the multiplication then the convolution of a stationary random function x(t) by two arbitrary functions h(t) and g(t) having Fourier transforms: $y(t) = [x(t).h(t)] * g(t)$. These relationships are applicable any time a recording made of the sum of a signal and of a noise is convoluted with a function g(t), that is, linearly smoothed by G(ν), the Fourier transform of g(t). In effect, it amounts to, whenever we wish to /41

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evaluate the effect of noise on the result, taking into account the fact that the duration of recording is not infinite. For this reason the result of convoluting the recorded noise, if it is obviously random, is, on the other hand, not stationary, and its statistical properties are a function of time. It then follows that the Wiener-Khinchine theorem [5] is not rigorously applicable. We will appraise the effect of the limitation on the duration of sampling to an interval T by replacing in the equations established in the paragraph above the function h(t) by its value:

$$h(t) = 1 \text{ for } -T/2 \leqslant t \leqslant +T/2$$
$$= 0 \text{ outside this range.}$$

Another application of these formulae comes up in the case where the recording of duration T is sampled in the form of equidistant discrete points. Such a sampling amounts to multiplying the recorded function by a function h(t) consisting of a "comb" of Dirac distributions. By applying the above-mentioned equations, it is the possible to evaluate, directly as a function of the sampling interval τ, the standard deviation $\sigma(\tau)$ of the noise after convolution with a function g(t). From this we deduce, as far as the noise is concerned, the optimum value of the sampling interval: it is the maximum value τ_o of τ for which the relative excess of $\sigma(\tau_o)$ with respect to $\sigma(o)$ can be considered acceptable. It is to be noted that such an evaluation is absolutely independent of the sampling theory or of the theorem of Shannon. The latter is only applicable, possibly, to the signal, and the maximum value of the sampling interval that it indicates simply represents an upper limit on τ that is not to be exceeded. These results are applied to the case where g(t) is a sinusoidal function.

In the framework of problems relative to measuring a signal, some of whose characteristics are known, we consider here a case where we have the following two data at our disposal: the signal is sinusoidal, the value of its period is known in advance. Moreover, this encompasses the limiting case of a DC signal. Because we are looking for an optimization, both of the results and of the ways of obtaining them, we are led to consider a fairly wide variety of methods and means, most of which are well known, and to achieve some sort of a synthesis. It seemed to us useful in this instance to recall and sometimes to specify a number of these questions by giving them a formulation as general as possible and by very frequently indicating the detail of the calculations that led to the results mentioned.

The plan of questions considered is given below. Let us specify the points which seem to be original: first of all, as it was said, section I, which considered the statistical properties of $y(t) = [x(t).h(t)] * g(t)$, with application to the case of sampling and of limited duration of observation; similarly, section V, relative to the probability densities $P(S')$ and $P(\phi')$, of the amplitude and of the phase, as they are obtained by the methods explained in the preceding section; finally, section VII, which is devoted to quantitative evaluation, for the case of smoothing by integrating over the limited duration followed by convolution with a sine wave, of the variation of the final signal-to-noise ratio as a function of the sampling interval and of the quantization step.

It is to be noted that the proposed methods for measuring the amplitude and the phase lead, either individually ot suitably combined, to the same result: the sinusoidal wave thus obtained is that which lends itself best, in the least-squares sense, to the recording of the signal and the noise. This does not necessarily mean, as it might, that the values of the amplitude and of the phase would be the best ones that we could obtain. Other methods could perhaps be considered, which, also using all available information, would lead to better or worse values of the signal-to-noise ratio. In this respect we mention the interesting approach proposed by W. B. McAdam [22]: based on a principle similar to that of doubly synchronous detection, it offers the two-fold advantage of using a fairly simple apparatus and of requiring only the recording of two quantities from which the values of the amplitude and of the phase are deduced. On the other hand, the signal-to-noise ratio thus obtained is less

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favorable by a factor of about 1 dB, it seems.

A direct application of these results comes up in the study of the brightness distribution of radio sources by interferometry with two antennae. For a given separation and orientation of the antennae the signal observed during the apparent rotation of the radio source studied produces in effect a very nearly sinusoidal shape whose period, defined by the horary coordinates of the center of the source, the geometry of the instrument and the wavelength of observation, varies slowly with time. The amplitude as well as the relative phase of this sinosoidal wave (relative to that of a point source located at the center of the source) also vary with time. We know that at a given instant they represent, at least as a first approximation, an element of the two dimensional Fourier transform of the brightness distribution of the observed radio source [13, 26, 27]. From the measurement of these quantities for various values of the hour angle, the distance between the antennae and the orientation of the base line, we can then deduce the inverse Fourier transform of the brightness distribution of the radio source.

Given that the observed signal is added to the noise, generally gaussian, due to the input stage of the receiver, the first problem that arises is to evaluate the phase and the relative amplitude of the sine wave with the best possible precision. Assuming that the useful duration of observation is known, that is, the time during which the amplitude and the relative phase of the signal do not vary noticeably, the signal remains to be converted into a sinosoidal wa of constant period (by a convenient modification of the timescale) for us to be able to apply the results given in the first portion o this article directly.

Such an application occured in the use of the two-antenna interfer ter with an East-West baseline and with variable displacement of Nançay station - an interferometer created essentially by E. LeRo and which J. LeQueux has used to study the brightness distribution right ascension of about forty radio sources [20]. The observatio were carried out in the vicinity of the meridian, in a time interv that could be as long

.. the hours, and it amounted to making use of such a recording time to improve also the sensitivity of the instrument. This way we have been led to building a device for summing the successive periods of the signal [32], and to establishing in this instance the theoretical results given in section III, VB and VI Bb (further details on the characteristics of this instrument can be found in references [19, 31, 21, 7, and reference 2, p. 309]).

Until now we have considered the problem of interpreting interferometric recordings essentially in terms of improving the signal-to-noise ratio by assuming that the characteristics of the signal are practically constant over a given time interval. In fact, as it has been mentioned, this signal is sinusoidal only as a first approximation, and, in addition, its amplitude and its relative phase vary continuously in time. It is therefore also necessary to evaluate the effect of the mathematical operators used on the signal in order to be able to determine the optimum duration of observation, that is, the maximum value of the time interval for which this distortion can be considered negligible.

It is to this two-fold aspect of the optimization of the interpretation of interferometric recordings, from the point of view of the signal and from the point of view of the noise, that the second part of this article will be devoted.

PLAN OF THE FIRST PART

Let the function $f_0(t) = s_0(t) + x_0(t)$, be defined from ∞ à $+\infty$, such that:

- $s_0(t)$ is a sinosoidal signal whose period T_0 is perfectly well known in advance (or, as a limiting case, is a DC signal);

- $x_0(t)$ is a stationary random noise of the order of two, whose statistical properties are known.

As this function is observable during a given time interval T, we seek to obtain the best possible measurement of the amplitude S and of the phase ϕ of the signal (that is, in effect, to make use of all information contained in this duration of observation), and to evaluate the degree of precision thus obtained. To this effect we will consider, separately and in the form of various combinations, the operations summarized in the system diagram below, and we will consider their effects on the precision of the final result.

The function $f_o(t)$ is observed during the time interval

$$t_0 - T/2 \leqslant t \leqslant t_0 + T/2.$$

That is, it has been multiplied by $h(t-t_o)$ such that

$$h(t-t_o) = 1 \text{ for } t_0 - T/2 \leqslant t \leqslant t_0 + T/2$$
$$= 0 \text{ outside of that interval,}$$

with the result that:

$$f_{t_0}(t) = f_0(t) \cdot h(t - t_0).$$

By various procedures this function $f_{t_o}(t)$ is linearly smoothed that is, convoluted with a function $g(t)$. From the fact that the duration of observation is finite, such an operation has the effect of transforming the initial noise into a random but non-stationary function whose statistical properties we have studied in section I. The results are specified in a completely general form, with $h(t-t_o)$ and $g(t)$ arbitrary.

The smoothing operation can be carried out by an electrical filter (block 2 of the system diagram, Figure 1), by a mathematical convolution (block 7), by an integration over the limited range (block 4), and more generally by a combination of two or more of these operations.

The summation (block 6), which consists of adding a number N of consecutive segments of width T_o, will also effect (section III) a linear smoothing of the noise, while preserving identically the shape of the signal when it is periodic with period T_o.

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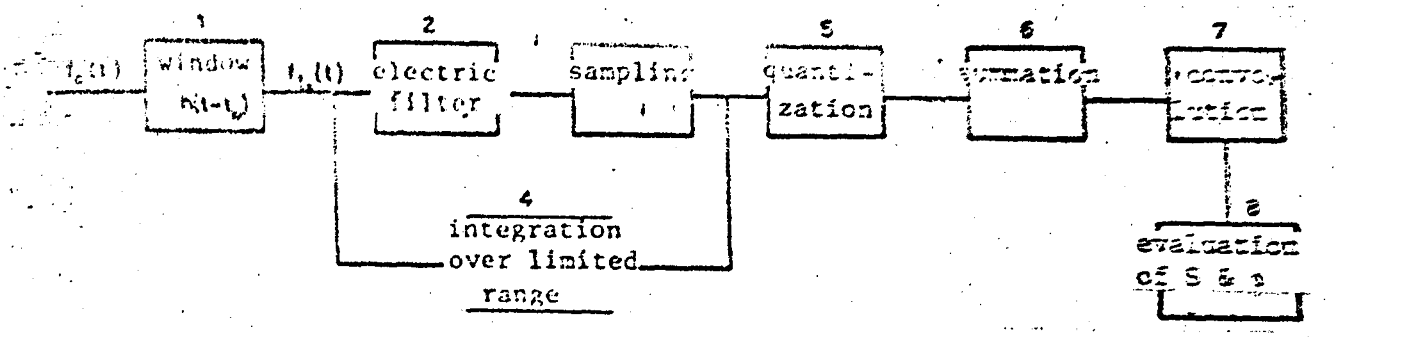

Figure 1. Block diagram of the operations considered.

In section II, we will study the properties of the convolution (or, which amounts to the same thing, of the cross-correlation) by

$$g(t) = \frac{2}{T}\cos 2\pi\nu_0 t,$$

where $\nu_0 = \frac{1}{T_0}$ is the frequency of the signal. The combination of such a filter and such a window of observation of duration T equal to an integer (or half-integer) multiple of T_0 is optimum, in that, on the one hand, it produces from the input signal an output signal which is identical and, on the other hand, it lends itself to making use of all the information contained in the recording. Hence the standard deviation of the noise after convolution is, at all points, very close to:

$$\sigma \doteq \sqrt{\frac{A_0}{T}}$$

where A_0 is the spectral energy density of the initial noise at the frequency ν_0. In effect the result $f_4(t)$ of the convolution of $f_3(t)$ with g(t) is a sinusoidal wave, the sum of $s_0(t)$ and of a sinusoidal function $y_4(t)$ of the same period and whose amplitude and phase are random. The evaluation of S and ϕ then consist of measuring the amplitude S' and the phase ϕ' of $f_4(t)$.

Section V is devoted to the evaluation of the statistical distribution laws of S' and of ϕ'.

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In section III the summation process will be studied.

We will show, or we will recall, in section IV, that the following operations are identical to a convolution of the recording of duration $T = NT_o$ with a $g(t) = \frac{2}{NT_0} \cos 2\pi\nu_0 t$:

a) The summation of N periods and the convolution of the result with $\frac{2}{T_0} \cos 2\pi\nu_0 t$.

b) Evaluation, for the harmonic $\nu_0 = \frac{1}{T_0}$, of the Fourier coefficients which identically represent $f_n(t)$ on the interval NT_o and mapped out by the corresponding sinusoidal wave.

c) The study of the sinusoidal wave of period T_o which best lends itself in the least-squares sense to the recording of $f_n(t)$.

d) The summation of N periods and the performance on this result of one or the other of the last two operations.

The result is, particularly for b), that the measurement of S' and of ϕ' becomes simply the evaluation of the Fourier coefficient for the harmonic ν_o.

Until now it has been assumed that the calculations are carried out on a continuous function and from values measured with infinite precision. In fact, the function is sampled in the manner of a series of discrete points at regular intervals τ, and the amplitude l_k of each of these points will be quantified, that is, represented by an integer L_K such that:

$$L_k - \frac{1}{2} \leqslant \frac{l_k}{q} < L_k + \frac{1}{2}$$

where q represents the width of the quantization steps.

The observed function is being sampled by means of n points at intervals τ, then is convoluted by a g(t):

$$g(t) = \frac{2}{n-1} \cos 2\pi \nu_0 t$$

The result is, like before, a sinusoidal wave of frequency ν_o, the sum of a sinusoidal signal and of a sinusoidal wave with random amplitude and phase. In order that the sinusoidal signal be identical to $s_o(t)$, it is sufficient that τ be a sub-multiple of $T = NT_c$, different from a whole number of half periods. As far as the noise is concerned, the sampling has the effect of increasing the standard deviation by a factor which is a function of τ and of the characteristics of the filter which precedes the sampling (blocks 2 or 4). In fact the sampling consists, even as the limitation of the duration of recording, of a multiplication of $f_o(t)$ by a function $h(t-t_o)$, more precisely, a "comb" of Dirac distributions. The conclusions of section I will therefore be directly applicable here. This question is the topic of section VI, where a certain number of results relative to quantization will be recalled. Moreover, even as the sampling, the quantization has the effect of introducing additional noise which is added to the initial noise $x_o(t)$ such that the standard deviation of this noise is also directly a function of the characteristics of the filter. In this same section VI, the study of the effect of the electric filter on the sampling and on the quantization will be preceded by some remarks on the third aspect of the role of such a filter: that of a possible substitute of, or complement to, any other form of smoothing. In particular we will see that the characteristics of the electric filter have practically no effect on the final signal-to-noise ratio of the result of the convolution of the recording with $g(t) = \frac{2}{T} \cos 2\pi \nu_0 t.$ Only the duration of observation T has any effect.

In section VII we will study the properties of "smoothing by integrating over a finite range" which combines the two functions of smoothing and sampling. This process offers the two-fold advantage over electric filters of not introducing a phase shift in the signal and of having a "temporary memory" strictly limited to the range of integration.

When the period of the signal is not known in advance, and provided that the duration of observation will be sufficiently long in view of the initial signal-to-noise ratio, the study of the amplitude and the phase can be carried out two ways:

a) Self-correlation of $f_L(t)$ which allows us to evaluate the period of the sinusoidal wave, as well as its amplitude, but with a high signal-to-noise ratio;

b) Convolution of $f_L(t)$ with a $g(t) = \frac{2}{T} \cos 2\pi\nu_0 t$ with possibly several trials for the values of ν_0 bracketing found in a).

The principle of self-correlation in the case of a sinusoidal wave in the presence of noise, as well as some qualitative considerations, will be recalled in section VIII.

In order not to render the notation difficult, we will consider each section to be independent. In particular, each time the expected operation is linear, by $f(t) = s(t) + x(t)$ we will denote the function on which it is performed, and by $l(t) = i(t) + y(t)$ the res of this operation.

I. FINITE DURATION OF OBSERVATION AND LINEAR SMOOTHING. EFFECT ON THE NOISE. /4

A. Linear Smoothing. The Duration of Observation is Infinite.

a. Let the signal $s_o(t)$ be an arbitrary function of time defined from ∞ to $+\infty$. If we apply it to a linear filter whose response to the Dirac distribution $\delta(t)$ (or, to percussion response), is $g(t)$, the resulting output function is

$$i_o(t) = \int_{-\infty}^{+\infty} s_o(\theta)\, g(t-\theta)\, d\theta \tag{1}$$

The convolution product of $s_o(t)$ with $g(t)$:

$$i_o(t) = s_o(t) * g(t). \tag{2}$$

When $s_o(t)$ has a Fourier transform:

$$S_o(\nu) = \int_{-\infty}^{+\infty} s_o(t)\, e^{2\pi i \nu t}\, dt$$

and if the same holds for $g(t)$, then equation (2) leads to

$$I_o(\nu) = S_o(\nu).G(\nu), \tag{3}$$

where $I_o(\nu)$ and $G(\nu)$ are the respective Fourier transforms of $i_o(t)$ /4 and $g(t)$. The frequency spectrum $S_o(\nu)$ of the function $s_o(t)$ has been smoothed by the frequency response $G(\nu)$ of the filter.

Let us recall that, by definition, a linear filter is an operator such that, if to a function $s_o(t)$ it gives a response $i_o(t)$, then the following two-fold condition is realized:

$$s_{01}(t) + s_{02}(t) \rightarrow i_{01}(t) + i_{02}(t)$$
$$s_o(t-\tau) \qquad \rightarrow i_o(t-\tau).$$

Here we consider linear filters in general. Electric filters are a particular class of these, characterized by the relation

$$g(t) = 0 \text{ for } t < 0.$$

b. In effect, the observed function is

$$f_0(t) = s_0(t) + x_0(t) \quad (4)$$

the sum of the signal $s_0(t)$ and of a particular sample of noise $x_0(t)$, a random stationary second order function. The filter $g(t)$ gives the response:

$$l_0(t) = i_0(t) + y_0(t) \quad (5)$$

with

$$y_0(t) = x_0(t) * g(t) \quad (6)$$

The random function $x_0(t)$ is stationary: its statistical properties are invariant with respect to any shift in the origin of time. In particular, it then follows that its average value

$$\overline{x_0(t)} = \frac{1}{T \to \infty} \int_{-T}^{+T} x_0(t)\, dt$$

and its correlation function

$$\rho(\tau) = \frac{1}{T \to \infty} \int_{-T}^{+T} x_0(t)\, x_0(t - \tau)\, dt$$

taken over a particular sample, are equal, respectively, to averages taken at a given instant t over an infinite number of samples.* Following that, we will evaluate these quantities in the form of averages taken with respect to time. The same property applies to the probability density that we will designate by $P(x)$.

This stationary random function is of the order two, that is, it is completely defined by the data on $x_0(t)$, $\rho(\tau)$, and $P(x)$. It will be assumed to be gaussian, of average value zero:

$$\overline{x_0(t)} = 0$$

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}.$$

* We can find a discussion of questions concerning random functions, in particular, in the chapter that A. Blanc-LaPierre has devoted to this topic in the book by A. Angot [5].

The standard deviation σ_o is defined by

$$\sigma_0^2 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} [x_0(t) - \bar{x}_0(t)]^2 \, dt$$

which leads to

$$\sigma_0^2 = \rho(0)$$

Finally, by $A(\nu)$ we designate the spectral energy density of $x_o(t)$. By definition, it is the Fourier transform of the correlation function:

$$A(\nu) = 4 \int_0^{\infty} \rho(\tau) \cos 2\pi \nu\tau \, d\tau. \tag{7}$$

Just like $\rho(\tau)$, $A(\nu)$ is a real and even function.

The response $y_o(t)$ that the filter $g(t)$ produces for $x_o(t)$ is also a stationary random second order function, and its average value is zero. Its spectral energy density $B(\nu)$ is, from the Wiener-Khinchin theorem, equal to the product of $A(\nu)$ and the square of the modulus of $G(\nu)$:

$$B(\nu) = A(\nu) . |G(\nu)|^2 \tag{8}$$

From this we deduce the correlation function $R(\tau)$ of $y_o(t)$:

$$R(\tau) = \int_0^{\infty} B(\nu) \cos 2\pi \nu\tau \, d\nu$$

as well as the standard deviation:

$$\sigma^2 = R(0) = \int_0^{\infty} A(\nu) \, |G(\nu)|^2 \, d\nu . \tag{9}$$

To summarize, when the signal $s_o(t)$ and the noise $x_o(t)$ are incident, from $-\infty$ to $+\infty$, on a linear filter with frequency response $G(\nu)$, it has the effect of:

- multiplying the spectrum of $s_o(t)$ by $G(\nu)$;
- multiplying the spectral energy density of $x_o(t)$ by $|G(\nu)|^2$.

B. Linear Smoothing. The Duration of Observation is Finite.

In practice, the duration of observation is limited, and the observed function is zero outside an interval

$$t_0 - \frac{T}{2} \leqslant t \leqslant t_0 + \frac{T}{2}.$$

This new function is represented in the form

$$f_{t_0}(t) = f_0(t)\, h(t - t_0) = s_{t_0}(t) + x_{t_0}(t) \tag{10}$$

with

$$h(t) = 1 \text{ for } -\frac{T}{2} \leqslant t \leqslant \frac{T}{2}$$
$$= 0 \text{ outside this range.}$$

The filter g(t) then produces the response

$$(11)\quad l_{t_0}(t) = f_{t_0}(t) * g(t)$$
$$= \int_{-\infty}^{+\infty} f_0(\theta)\, h(\theta - t_0)\, g(t - \theta)\, d\theta = i_{t_0}(t) + y_{t_0}(t), \tag{11}$$

with:

$$i_{t_0}(t) = [s_0(t) . h(t - t_0)] * g(t) \tag{12}$$
$$y_{t_0}(t) = [x_0(t) . h(t - t_0)] * g(t) \tag{13}$$

By taking the Fourier transforms of the two terms of Equation (12), we have a new expression of the spectrum of the output signal

$$I_{t_0}(\nu) = [S_0(\nu) * H(\nu)\, e^{2\pi i \nu t_0}] . G(\nu) \tag{14}$$

where H(ν) is the Fourier transform of h(t).

---

* For all that refers to the Fourier transform of functions and of distributions, the reader could consult the work of J. Arsac [2] which also treats the case of random functions.

Because the duration of observation is limited to the window $h(t-t_0)$, $s_0(t)$ and $x_0(t)$ are no longer linearly smoothed. Consequently, as far as the noise is concerned, the random function $y_{t_0}(t)$ is not stationary: the statistical properties of the quantities $y_{t_0}(t)$, taken at a given instant t over an infinite number of simultaneous trials, are a function of t.

Let $y_{t_0}(t = t_0 + \alpha)$ be the value of $y_{t_0}(t)$ at a distance α from the origin t_0:

$$y_{t_0}(t_0 + \alpha) = \int_{-\infty}^{+\infty} x_0(\theta)\, h(\theta - t_0)\, g(t_0 + \alpha - \theta)\, d\theta. \tag{15}$$

We seek to evaluate the average value and the standard deviation of $y_{t_0}(t_0 + \alpha)$. Given that the random function $x_0(t)$ is stationary, this amounts to carrying out these averages:

- over an infinite number of trials obtained by applying equation (15) to an infinite number of samples $x_0(t)$, α, and t_0 being constant;

- or over an infinite number of trials obtained by displacing the window $h(t-t_0)$ along a sample $x_0(t)$, while t_0 varies between $-\infty$ and $+\infty$, and α remains constant.

It is this second aspect that we will consider. The result of an operation so defined is (Figure 2) that a new function $y_\alpha(t)$ corresponds to the function $x_0(t)$ such that:

$$y_\alpha(t = t_0 + \alpha) = y_{t_0}(t_0 + \alpha) \tag{16}$$

or, from Equation (15),

$$y_\alpha(t = t_0 + \alpha) = \int_{-\infty}^{+\infty} x_0(\theta)\, h(\theta - t_0)\, g(t_0 + \alpha - \theta)\, d\theta$$

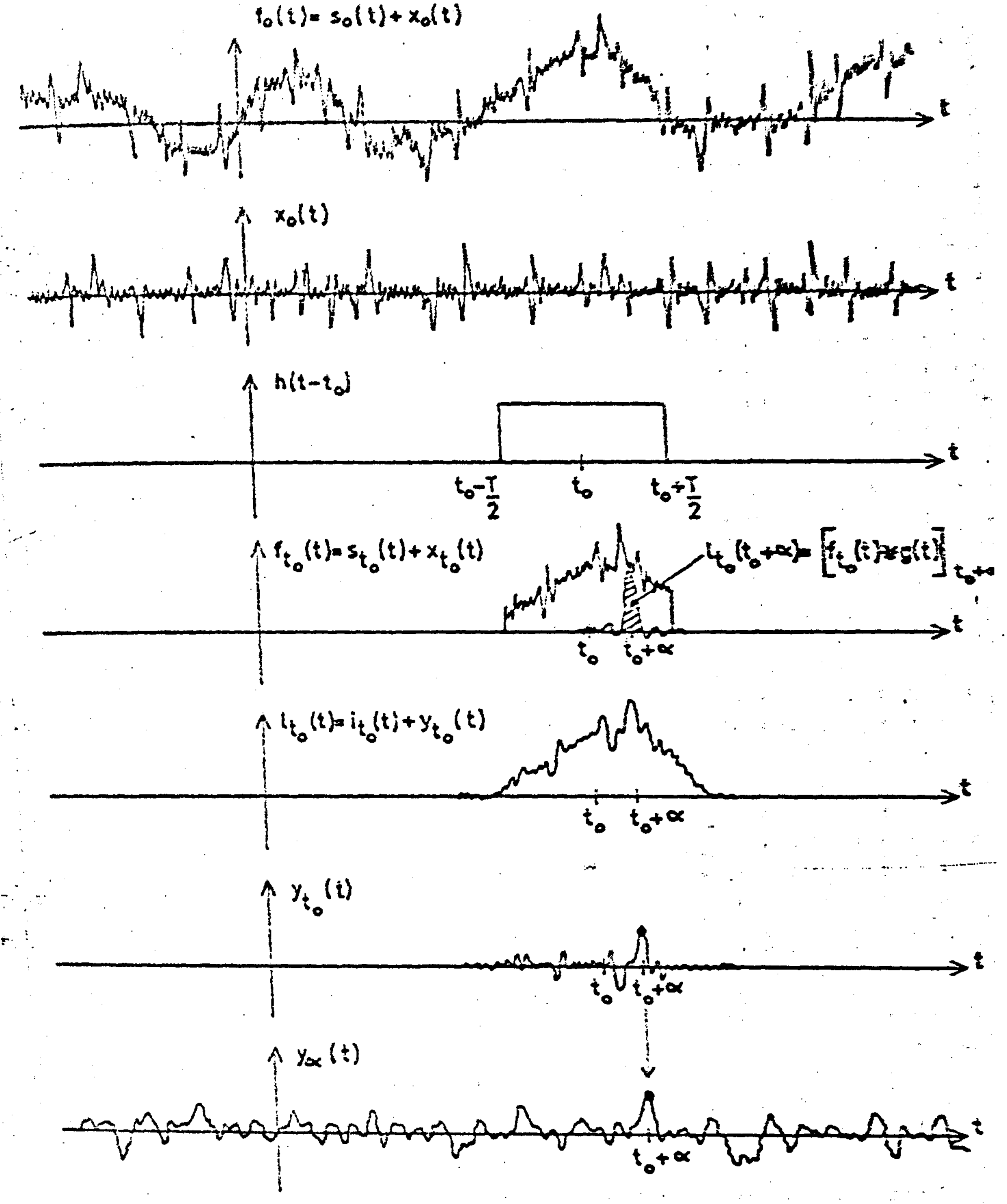

Figure 2. Limited duration of observation and linear smoothing. In order to evaluate the statistical properties $y_{t_0}(t_0+\alpha)$ we consider the ensemble of $y_\alpha(t)$ of trials obtained by displacing the window $h(t-t_0)$ between $-\infty$ and $+\infty$.

whose particular value for $t = t_0 + \alpha$ is:

$$y_\alpha(t) = \int_{-\infty}^{+\infty} x_0(\theta)\, h(\theta - t + \alpha)\, g(t - \theta)\, d\theta$$

This equation represents the convolution product:

$$y_\alpha(t) = x_0(t) * [h(-t + \alpha).g(t)] \qquad (17)$$

The function $y_\alpha(t)$ is thus obtained from the noise $x_0(t)$ by linearly smoothing the gain:

$$J_\alpha(\nu) = [H(-\nu)\, e^{2\pi i \nu \alpha}] * G(\nu) \qquad (18)$$

the Fourier transform of

$$[h(-t + \alpha).g(t)].$$

It is, even as $x_0(t)$ is, a stationary random second order function, and whose distribution is gaussian. It has the average value:

$$\overline{y_\alpha(t)} = 0 \qquad (19)$$

and a standard deviation σ_α such that:

$$\sigma_\alpha^2 = \int_0^\infty A(\nu)|J(\nu)_\alpha|^2\, d\nu . \qquad (20)$$

where $A(\nu)$ is the spectral energy density of $x_0(t)$.

In summary, the standard deviation σ_α of the fluctuations which at the point $t = t_0 + \alpha$ are superposed on the signal $i_\alpha(t)$ whose spec-

$$I_\alpha(\nu) = [S_0(\nu) * H(\nu)\, e^{2\pi i \nu \alpha}].G(\nu) \qquad (14)$$

is given by

$$\boxed{\begin{array}{l} \sigma_\alpha^2 = \int_0^\infty A(\nu)|J_\alpha(\nu)|^2\, d\nu \\ \text{with:} \\ J_\alpha(\nu) = [H(-\nu)\, e^{2\pi i \nu \alpha}] * G(\nu) \end{array}} \qquad (21)$$

In the present case, the function h(t) which multiplies $f_o(t)$ is a window of width T and its Fourier transform has the value

$$H(\nu) = \frac{\sin \pi\nu T}{\pi\nu}. \tag{22}$$

As T approaches infinity, H(ν) approaches the Dirac distribution δ(ν) and we are led to the equations of the preceding section.

The correlation coefficient $C_\alpha(\tau)$ remains to be evaluated between the two values $y_L(t_0 + \alpha)$ and $y_L(t_0 + \alpha + \tau)$ which are separated by the interval τ:

$$C_\alpha(\tau) = \frac{R_\alpha(\tau)}{R_\alpha(0)}.$$

with

$$R_\alpha(\tau) = \overline{y_L(t_0 + \alpha)\, y_L(t_0 + \alpha + \tau)}.$$

We obtain (see appendix):

$$\boxed{R_\alpha(\tau) = \frac{1}{2}\int_{-\infty}^{+\infty} A(\nu)\, J_\alpha(\nu)\, J^*_{\alpha+\tau}(\nu)\, e^{2\pi i\nu\tau}\, d\nu}. \tag{A-9}$$

thus being led, by a different approach, to Equation (20):

$$R_\alpha(0) = \sigma_\alpha^2 = \int_0^\infty A(\nu)|J(\nu)_\alpha|^2\, d\nu.$$

C. Application to the Case of a Sampling.

The general equations above are applied to the case of a sampling (Figure 3) by which the distribution:

$$f_L(t) = \sum_{k=0}^{n-1} f_0(t_0 + k\tau)\, \delta[t - (t_0 + k\tau)] \tag{2}$$

corresponds to $f_o(t)$, with

$$T = (n-1)\tau \tag{2}$$

In effect, we can then write Equation (23) in the form:

$$f_e(t) = \sum_{k=0}^{n-1} f_0(t)\,\delta[t-(t_0+k\tau)].$$

or

$$f_e(t) = f_0(t).h(t-t_0).$$

with

$$h(t) = \sum_{k=0}^{n-1} \delta[t-k\tau]. \qquad (25)$$

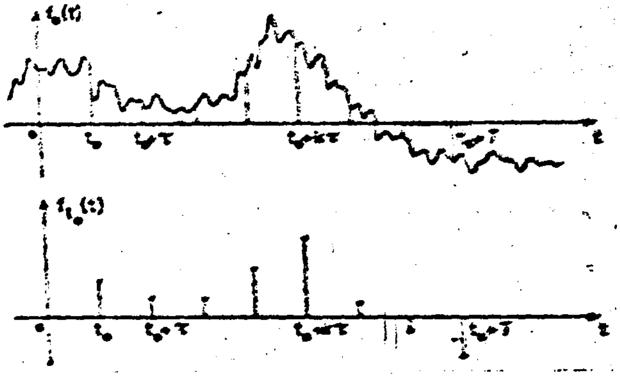

Figure 3. Sampling.

The Fourier transform of h(t) is:

$$H(\nu) = \sum_{k=0}^{n-1} e^{2\pi i\nu k\tau}.$$

This is the sum of a geometric progression:

- with the first term: $a = 1$
- with argument: $r = e^{2\pi i\nu\tau}$
- with last term: $m = e^{2\pi i\nu(n-1)\tau}$,

and it has the value:

$$H(\nu) = \frac{mr-a}{r-1}$$

or

$$H(\nu) = \frac{\sin \pi\nu n\tau}{\sin \pi\nu\tau}\, e^{i\pi\nu(n-1)\tau}. \qquad (26)$$

By replacing $H(\nu)$ by this value in Equation (21), we can then evaluate σ_α as a function of the width τ of the sampling interval. By this means it is then possible to determine the optimum interval of sampling, as far as the noise is concerned: it is the largest value τ_0 of τ for which the relative increase of σ_α remains acceptable. This is applied in section VI to the case of the convolution with a sinusoidal waveform.

II. CONVOLUTION WITH A SINUSOIDAL WAVE OF THE SAME PERIOD.

A. Convolution and Cross-Correlation.

The cross-correlation product of $f_o(t)$ with $g(t)$ is defined by (see, for example, ref. [16]):

$$k(t) = \int_{-\infty}^{+\infty} f_o(t+\theta)\, g(\theta)\, d\theta,$$

or, replacing $(t+\theta)$ by θ',

$$k(t) = \int_{-\infty}^{+\infty} f_o(\theta')\, g(\theta' - t)\, d\theta'$$

In the case where g(t) is even, this equation represents the convolution product:

$$k(t) = f_o(t) * g(t) = 1(t).$$

In the following, we will consider:

$$g(t) = \frac{2}{T} \cos 2\pi \nu_o t,$$

and there will be no reason to distinguish between cross-correlatio and convolution. We will use the term "convolution" to designate o or the other of these operations.

B. Duration of Observation T. * Convolution by $g(t) = \frac{2}{T} \cos 2\pi \nu_o t$.

Let

$$l(t) = [f_o(t) . h(t - t_o)] * \frac{2}{T} \cos 2\pi \nu_o t \qquad (27$$

be the result of convolution of an arbitrary function $f_o(t)$ observe in the interval

$$t_o - T/2 \leq t \leq t_o + T/2,$$

---

* In order to simplify the notation, we will omit the index t the following discussion, as regards the input and output function $f_o(t)$ and $l_o(t)$.

with the sinusoidal wave

$$g(t) = \frac{2}{T} \cos 2\pi \nu_0 t \qquad (28)$$

Equation (27) can be written as:

$$l(t) = \frac{2}{T} \int_{t_0-T/2}^{t_0+T/2} f_0(\theta) \cos 2\pi \nu_0 (t-\theta)\, d\theta$$

or as

$$l(t) = \left[\frac{2}{T} \int_{t_0-T/2}^{t_0+T/2} f_0(\theta) \cos 2\pi \nu_0 \theta \, d\theta\right] \cos 2\pi \nu_0 t + \left[\frac{2}{T} \int_{t_0-T/2}^{t_0+T/2} f_0(\theta) \sin 2\pi \nu_0 \theta \, d\theta\right] \sin 2\pi \nu_0 t \qquad (29)$$

which is of the form

$$l(t) = a' \cos 2\pi \nu_0 t + b' \sin 2\pi \nu_0 t. \qquad (30)$$

The function l(t) is sinusoidal with period $T_o = \frac{1}{\nu_o}$, the coefficients a' and b' are the coefficients for the ν_o term of the Fourier series which identically represents $f_o(t)$ in the interval

$$t_0 - T/2 \leqslant t \leqslant t_0 + T/2.$$

In particular, this is applied to

$$f_0(t) = s_0(t) + x_0(t)$$

the sum of the sinusoidal wave

$$\begin{aligned} s_0(t) &= S \cos (2\pi \nu_0 t + \varphi) \\ &= a \cos 2\pi \nu_0 t + b \sin 2\pi \nu_0 t \end{aligned} \qquad (31)$$

and of the noise $x_o(t)$.

a) Effect on the signal.

By applying, for example, Equation (29) to $s_o(t)$, we obtain:

$$i(t) = S\sqrt{1 + \left(\frac{\sin 2\pi \nu_0 T}{2\pi \nu_0 T}\right)^2 + 2\left(\frac{\sin 2\pi \nu_0 T}{2\pi \nu_0 T}\right)\cos(4\pi \nu_0 t_0 + 2\varphi)}\cos(2\pi \nu_0 t + \psi) \quad (32)$$

with

$$\psi = \operatorname{Arc\,tg} \frac{\sin\varphi - \frac{\sin 2\pi \nu_0 T}{2\pi \nu_0 T}\sin(4\pi \nu_0 t_0 + \varphi)}{\cos\varphi + \frac{\sin 2\pi \nu_0 T}{2\pi \nu_0 T}\cos(4\pi \nu_0 t_0 + \varphi)}.$$

When T is arbitrary, the amplitude and the phase of i(t) are functions of t_o, T and ϕ. It is to be noted that in terms of the relationships which exist between these two parameters, the amplitud can equally well be larger or smaller than S, even as the standard deviation of the noise y(t) depends, at each point, only on T (section b). In fact, when T is sufficiently large relative to t_o (of the order of several t_o) the quantity $\sin 2\pi\nu_0 T / 2\pi\nu_0 T$ becomes very small and the amplitude and the phase of i(t) approach those of s_o(
In the particular case where T is a multiple of $T_o/2$, the result of the convolution is identical to the initial sinusoidal wave $s_o(t)$:

$$i(t) = s_0(t) = S\cos(2\pi \nu_0 t + \varphi). \quad (33)$$

Later on, we will take $T + NT_o$, where N is an arbitrary integer.

b) Effect on the noise.

Onto the signal i(t) is superimposed the function

$$y(t) = \left[\frac{2}{T}\int_{t_0-T/2}^{t_0+\frac{T}{2}} x_0(\theta)\cos 2\pi \nu_0 \theta \, d\theta\right]\cos 2\pi \nu_0 t + \left[\frac{2}{T}\int_{t_0-T/2}^{t_0+T/2} x_0(\theta)\sin 2\pi \nu_0 \theta \, d\theta\right]\sin 2\pi \nu_0 t,$$

sinusoidal with the same period as the signal, and whose amplitude and phase are random.

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At any point $t = t_o + \alpha$, the average value of $y(t)$ is zero. The standard deviation σ_α is evaluated from Equation (21) by replacing $G(\nu)$ by its value

$$G(\nu) = \frac{1}{T}\{\delta(\nu + \nu_0) + \delta(\nu - \nu_0)\}. \qquad (35)$$

It becomes:

$$\begin{aligned} J_\alpha(\nu) &= \left[\frac{\sin \pi\nu T}{\pi\nu} e^{2\pi i\nu\alpha}\right] \\ &\qquad * \frac{1}{T}\{\delta(\nu + \nu_0) + \delta(\nu - \nu_0)\} \\ &= \frac{\sin \pi(\nu + \nu_0)T}{\pi(\nu + \nu_0)T} e^{2\pi i(\nu+\nu_0)\alpha} \\ &\qquad + \frac{\sin \pi(\nu - \nu_0) T}{\pi(\nu - \nu_0) T} e^{2\pi i(\nu-\nu_0)\alpha}. \end{aligned} \qquad (36)$$

After evaluation of $|J_\alpha(\nu)|^2$, the equation

$$\sigma_\alpha^2 = \int_0^\infty A(\nu)|J_\alpha(\nu)|^2 \, d\nu, \qquad (20)$$

becomes

$$\boxed{\begin{aligned} \sigma_\alpha^2 = \int_0^\infty A(\nu) \Bigg[&\left(\frac{\sin \pi(\nu + \nu_0) T}{\pi(\nu + \nu_0) T}\right)^2 \\ &+ \left(\frac{\sin \pi(\nu - \nu_0) T}{\pi(\nu - \nu_0) T}\right)^2 \\ &+ 2 \cos 4\pi\nu_0 \alpha \frac{\sin \pi(\nu + \nu_0) T \sin \pi(\nu - \nu_0) T}{\pi(\nu + \nu_0) T . \pi(\nu - \nu_0) T} \Bigg] d\nu \end{aligned}} \qquad (37)$$

Let us recall that $A(\nu)$ is the spectral energy density of $x_o(t)$.

We put:

$$\sigma_\alpha^2 = I + I_\alpha,$$

with

$$I = \int_0^\infty A(\nu) \left[\left(\frac{\sin \pi(\nu + \nu_0) T}{\pi(\nu + \nu_0) T}\right)^2 + \left(\frac{\sin \pi(\nu - \nu_0) T}{\pi(\nu - \nu_0) T}\right)^2\right] d\nu, \qquad (38)$$

and

$$I_\alpha = 2 \cos 4\pi \nu_0 \alpha \int_0^\infty A(\nu) \frac{\sin \pi(\nu + \nu_0) T . \sin \pi(\nu - \nu_0) T}{\pi(\nu + \nu_0) T . \pi(\nu - \nu_0) T} d\nu. \qquad (39)$$

In the case where $A(\nu)$ is constant (white noise), $A(\nu) = A_0$, and we have

$$I = \frac{A_0}{T} \tag{40}$$

and, for $T \neq NT_0$ (as we are assuming here),

$$I_\alpha = 0 \tag{41}$$

This last result is easily established by noting that the integral of Equation (39) is the scalar product of two functions; its value is therefore equal to the scalar product of the Fourier transform of one and the conjugate of the Fourier transform of the other, which leads to:

$$I_\alpha = 2 \cos 4\pi \nu_0 \alpha \int_{-\infty}^{+\infty} h(t)\, e^{2\pi i \nu t} . h(t)\, e^{2\pi i \nu t}\, dt$$

$$= \cos 4\pi\, \nu_0\, \alpha \int_{-\frac{NT_0}{2}}^{+\frac{NT_0}{2}} e^{4\pi i \nu t}\, dt = 0.$$

In the general case, the quantity I_α remains very small compar to I. For example, when $A(\nu)$ is symmetrical in $\partial\nu_0$, and has its maximum at $\nu = \nu_0$, we can easily estimate I_α by:

$$|I_\alpha| < \frac{A_0}{10\, N^2\, T_0} \tag{42}$$

by writing

$$I_\alpha = 2 \cos 4\pi\, \nu_0\, \alpha (I_1 + I_2),$$

with

$$I_1 = \int_0^{2\nu_0} A(\nu) \frac{\sin \pi(\nu + \nu_0)\, T . \sin \pi(\nu - \nu_0)\, T}{\pi(\nu + \nu_0)\, T . \pi(\nu - \nu_0)\, T}\, d\nu$$

$$= 2 \int_0^{\nu_0} A(\nu) \frac{\sin^2 \pi \nu T}{\pi^2\, T^2 (\nu^2 - 4\nu_0^2)}\, d\nu$$

$$I_1 < 2\, A_0 \int_0^{\nu_0} \frac{1}{\pi^2\, T^2 (\nu^2 - 4\nu_0^2)}\, d\nu,$$

and

$$I_2 = \int_{2\nu_0}^{\infty} A(\nu) \frac{\sin \pi(\nu + \nu_0)\, T . \sin \pi(\nu - \nu_0)\, T}{\pi(\nu + \nu_0)\, T . \pi(\nu - \nu_0)\, T}\, d\nu$$

$$< A_0 \int_{2\nu_0}^{\infty} \frac{1}{\pi^2\, T^2 (\nu^2 - \nu_0^2)}\, d\nu$$

Comparing Equations (40) and (42), we have

$$\frac{|I_s|}{I} < \frac{1}{10\,N} \tag{43}$$

Although, in all rigor, the standard deviation should be a function of the distance α from the origin t_0, we can write, with an approximation which is better as N is larger (and rigorously true when $A(\nu)$ is constant):

$$\sigma_a^2 \doteq \sigma^2 = \int_0^{\infty} A(\nu)\left[\left(\frac{\sin \pi(\nu+\nu_0)T}{\pi(\nu+\nu_0)T}\right)^2 + \left(\frac{\sin \pi(\nu-\nu_0)T}{\pi(\nu-\nu_0)T}\right)^2\right] d\nu, \tag{44}$$

an equation that can also be written as:

$$\boxed{\sigma_a^2 \doteq \sigma^2 = \int_{-\infty}^{+\infty} A(\nu)\left(\frac{\sin \pi(\nu-\nu_0)T}{\pi(\nu-\nu_0)T}\right)^2 d\nu}, \tag{45}$$

by putting

$$A(-\nu) = A(\nu).$$

The function

$$|J(\nu)|^2 = \left(\frac{\sin \pi(\nu-\nu_0)T}{\pi(\nu-\nu_0)T}\right)^2. \tag{46}$$

is characterized by:

- a main lobe centered on $\nu = \nu_0$, with height 1 and width between its zeroes equal to 2/T.

- a total area equal to 1/T, the area under the secondary lobes being less than 1/5 T.

In Figure 4 the hatched area represents

$$\boxed{\sigma^2 \doteq \frac{A_0}{T}}. \tag{47}$$

while the area $\sigma_0^2 = \int_0^{\infty} A(\nu)\, d\nu$ is equal to the square of the standard deviation of the noise $x_0(t)$ on the recording.

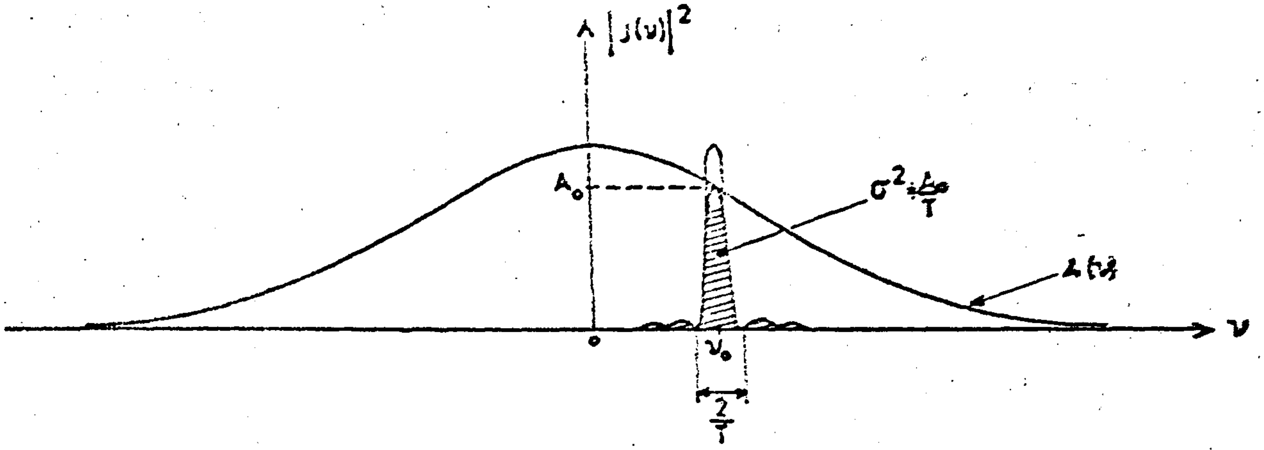

Figure 4. Noise energy after convolution with a sinusoidal wave. Practically independent of α, it is very closely approximated by the hatched area.

The improvement in the signal-to-noise ratio is therefore:

$$g = \frac{S/\sigma}{S/\sigma_0}, \tag{48}$$

which we can evaluate graphically. It is of the order of:

$$g \simeq \frac{\sqrt{A_0 B}}{\sqrt{A_0/T}} = \sqrt{BT}, \tag{49}$$

where B represents the bandpass of the filter that precedes the recording.

Comparison of Equations (9) and (45) shows that the convolutio of a recording duration T with a $g(t) = (2/T) \cos 2\pi \nu_0 t$ has practic ly the same effect on the noise as a filter with energy gain $|J(\nu)|^2$ to which this noise would be applied for an infinite length of time

On the other hand, Equation (47) makes it apparent that, only from the point of view of the standard deviation σ, everything hap- pens as if it went through a rectangular filter with a bandpass eq to 1/T.

It is to be noted, as will be recalled in section VI, that th final signal-to-noise ratio is practically independent from the characteristics of the electric filter which precedes the recordin

The effect of such a filter, the frequency gain $G_2(\nu)$, is, in effect, to multiply by the same factor $|G_2(\nu_0)|$:

- the amplitude of the sinusoidal wave;
- and the square root of the spectral energy density of the noise at the frequency ν_0 (hence, from Equation (47), the final standard deviation of the noise).

As far as the correlation coefficient $C_\alpha(\tau)$ between two values of y(t) separated by τ are concerned, we can show that it is, just like σ_α^2, practically independent of α and has the value (see appendix):

$$\boxed{C_\alpha(\tau) \doteq C(\tau) = \cos 2\pi\nu_0\tau}. \tag{A-12}$$

C. Determination of the Amplitude and the Phase of the Signal.

Since the result of the convolution with a $g(t) = (2/T) \cos 2\pi\nu_0 t$ is a sinusoidal wave of the same period as the signal, and whose amplitude S' and the phase ϕ' are random quantities whose average values approach S and ϕ' (when the signal-to-noise ratio is high), the evaluation of these two latter quantities consists of measuring S' and ϕ'. The study of the statistical properties of S' and ϕ' is the topic of section V.

D. Remark: Integration of a DC Signal.

There is an analogy between the convolution of a sinusoidal wave in the case of a periodic signal and the integration in the case of a DC signal. The latter operation corresponds, in effect, to the limiting case of convolution when ν_0 approaches zero, as Equation (28) is replaced by g(t) = 1/T. As we will see in section VII, Equations (37) and (47) then become, respectively, $\sigma^2 = \int_0^\infty A(\nu) \left(\frac{\sin \pi\nu T}{\pi\nu T}\right)^2 d\nu$

and

$$\sigma^2 \doteq \frac{A_0}{2T}$$

where A_0 this time is the spectral energy density of the noise at frequency zero.

III. SUMMATION.

This process of improving the signal-to-noise ratio used mostly in radar [12, 14] is applicable any time we deal with a periodic signal. We have used it [32] for interferometric observations in radio astronomy, since the signal is sinusoidal. We will find in the above-mentioned article, as well as in references [19 and 21], some examples of the degree of improvement that it allows us to obtain.

Since the function f(t) = s(t) + x(t), defined over the interva[l] $t_0 \leqslant t \leqslant t_0 + NT_0$ and zero outside this interval, is divided into N consecutive segments of width T_0, by summation we refer to the oper[a]tion which consists of adding these N segments among each other (Figure 5).

The result of the summation is written as

$$l(t) = \frac{1}{N} \sum_{k=0}^{N-1} f(t + kT_0), \tag{50}$$

with

$$t_0 \leqslant t \leqslant t_0 + T_0.$$

This equation expresses the fact that, corresponding to the part f(t) of $f_0(t)$ defined from $-\infty$ to $+\infty$:

$$f(t) = f_0(t)\, h_1(t - t_0)$$

with

$h_1(t) = 1$ for $0 \leqslant t \leqslant NT_0$
= 0 outside this range,

there is the part l(t) of $l_0(t)$ also defined from $-\infty$ to $+\infty$:

$$l(t) = l_0(t)\, h_2(t - t_0)$$

with

$h_2(t) = 1$ for $0 \leqslant t \leqslant T_0$
= 0 outside this range,

such that the function $l_0(t)$ is related to $f_0(t)$ by the equation:

$$l_o(t) = \frac{1}{N}\sum_{k=0}^{N-1} f_o(t + kT_o). \qquad (51)$$

From the study of $l_o(t)$ we deduce the properties of $l(t)$. Equation (51) can be written as

$$l_o(t) = \frac{1}{N}\sum_{k=0}^{N-1} f_o(t) * \delta(t + kT_o),$$

which is the convolution product /42

$$l_o(t) = f_o(t) * \frac{1}{N}\sum_{k=0}^{N-1} \delta(t + kT_o). \qquad (52)$$

The function $l_o(t)$ is thus obtained from $f_o(t)$ by the linear smoothing of gain:

$$G(\nu) = \frac{1}{N}\sum_{k=0}^{N-1} e^{-2i\pi\nu kT_o}, \qquad (53)$$

which is the Fourier transform of

$$g(t) = \frac{1}{N}\sum_{k=0}^{N-1} \delta(t + kT_o).$$

that is,

$$\boxed{G(\nu) = \frac{\sin \pi\nu N T_o}{N \sin \pi\nu T_o} e^{-i\pi\nu(N-1)T_o}} \qquad (54)$$

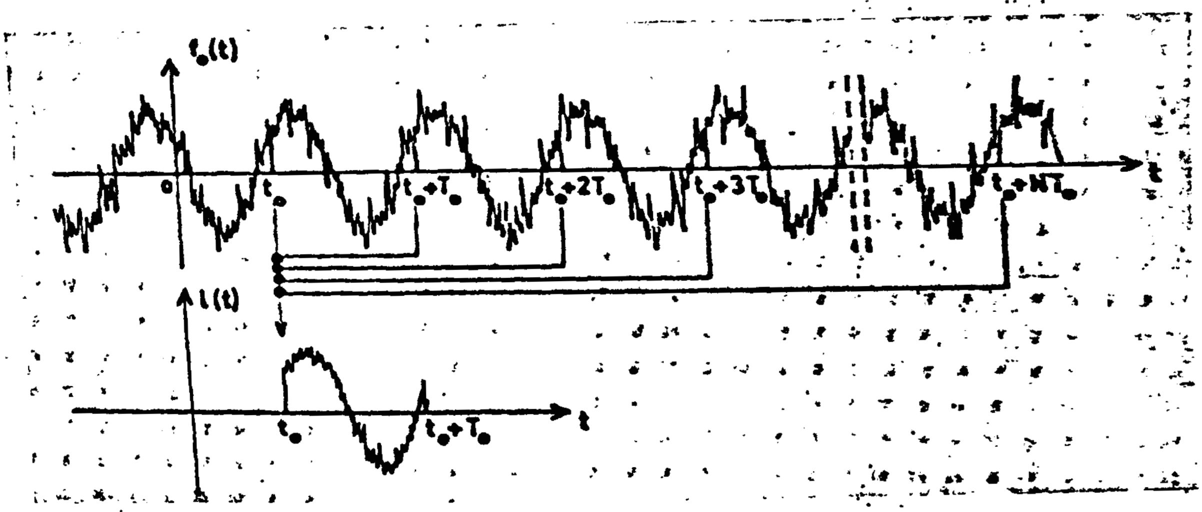

Figure 5. Summation principle.

A. <u>Effect on the Signal.</u>

As far as the signal is concerned, we have

$$I_o(\nu) = S_o(\nu).G(\nu),$$

with

$$S_o(\nu) = \frac{S}{2} e^{i\varphi} \delta(\nu + \nu_o) + \frac{S}{2} e^{-i\varphi} \delta(\nu - \nu_o) \tag{55}$$

which leads to

$$I_o(\nu) = \left(\frac{\sin \pi\nu_o N T_o}{N \sin \pi\nu_o T_o}\right) \left[\frac{S}{2} e^{i\varphi} e^{i\pi\nu_o(N-1)T_o} \delta(\nu + \nu_o) + \frac{S}{2} e^{-i\varphi} e^{-i\pi\nu_o(N-1)T_o} \delta(\nu - \nu_o)\right]. \tag{56}$$

When T_o, the width of the elementary segment is equal to $1/\nu_o$, the period of the sinusoidal wave, then:

$$I_o(\nu) = \frac{S}{2} e^{i\varphi} \delta(\nu + \nu_o) + \frac{S}{2} e^{-i\varphi} \delta(\nu - \nu_o) = S_o(\nu),$$

and we have

$$i(t) = i_o(t).h_2(t - t_o) = S \cos(2\pi\nu_o t + \varphi), \tag{57}$$

with

$$t_o \leq t \leq t_o + T_o.$$

We rederive the result, evident <u>a priori</u>, that the addition of N periods of a sinusoidal wave in phase leads, after division by N, to an identical period of this sinusoidal wave.

If T_o is seen to be different from $1/\nu_o$, the function $i_o(t)$ is sinusoidal wave with an amplitude reduced by the ratio of $\sin \pi\nu_o N T_o / N \sin \pi\nu_o T_o$ and with phase equal to:

$$[\varphi + \pi\nu_o(N - 1) T_o].$$

In addition, the part of $i(t)$ of $i_o(t)$ is not equal any longer to a period of $s_o(t)$.

B. Effect on the Noise.

Applied to noise, Equation (52) is written as:

$$y_0(t) = x_0(t) * g(t).$$

The function y(t):

$$y(t) = \frac{1}{N} \sum_{k=0}^{N-1} x_0(t + kT),$$

with:

$$t_0 \leqslant t \leqslant t_0 + T_0$$

is part of $y_0(t)$ taken over the interval $t_0 \leqslant t \leqslant t_0 + T_0$. Its statistical properties are, therefore, those of $y_0(t)$.

The spectral energy density of $y_0(t)$ is

$$B(\nu) = A(\nu).|G(\nu)|^2, \qquad (8)$$

with

$$|G(\nu)|^2 = \left(\frac{\sin \pi\nu N T_0}{N \sin \pi\nu T_0}\right)^2. \qquad (58)$$

From this we deduce its correlation function

$$R(\tau) = \int_0^\infty B(\nu) \cos 2\pi \nu\tau \, d\nu,$$

that is,

$$R(\tau) = \int_0^\infty A(\nu) \left(\frac{\sin \pi\nu N T_0}{N \sin \pi\nu T_0}\right)^2 \cos 2\pi \nu\tau \, d\nu, \qquad (59)$$

and its standard deviation σ_N:

$$\boxed{\sigma_N^2 = R(0) = \int_0^\infty A(\nu) \left(\frac{\sin \pi\nu N T_0}{N \sin \pi\nu T_0}\right)^2 d\nu} \qquad (60)$$

The function $|G(\nu)^2|$, which is periodic, includes a series of main lobes:

- at intervals of $1/T_0$
- of base $2/NT_0$
- of height 1

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- of total area, including the related secondary lobes, equal to $1/NT_0$.

In Figure 6 we have shown the product of $|G(\nu)|^2$ and $A(\nu)$. As the hatched area is proportional to σ_N^2 and the total area is proportional to σ_0^2 (square of the standard deviation of the noise before summation) the improvement in signal-to-noise ratio due to the summation:

$$g = \frac{S/\sigma_N}{S/\sigma_0}, \tag{61}$$

can be evaluated very simply from this figure. In general, it is approximately:

$$g \simeq \sqrt{N}. \tag{62}$$

The minimum value of σ_N^2:

$$(\sigma_N^2)_m \doteqdot \frac{A_0}{N T_0}, \tag{63}$$

is obtained when the filter that precedes the summation has a band-pass B less than or equal to $1/T_0$ and centered at $\nu_0 = 1/T_0$.

These results have been obtained initially (Vinokur, 1959) by directly evaluating the correlation function $R(\tau)$ of:

$$y(t) = \frac{1}{N}\sum_{k=0}^{N-1} x_0(t + kT).$$

From Equation (59) we then deduce the spectral energy density of the noise after summation

$$B(\nu) = A(\nu)\left(\frac{\sin \pi\nu N T_0}{N \sin \pi\nu T_0}\right)^2, \tag{64}$$

thus rederiving the Wiener-Khintchine theorem:

$$B(\nu) = A(\nu).|G(\nu)|^2,$$

in this particular case.

In this form, the calculations are fairly long, and J. Arsac [2, p. 309] has shown that this question could be treated more simply and more logically by starting from Equation (52). In the book by

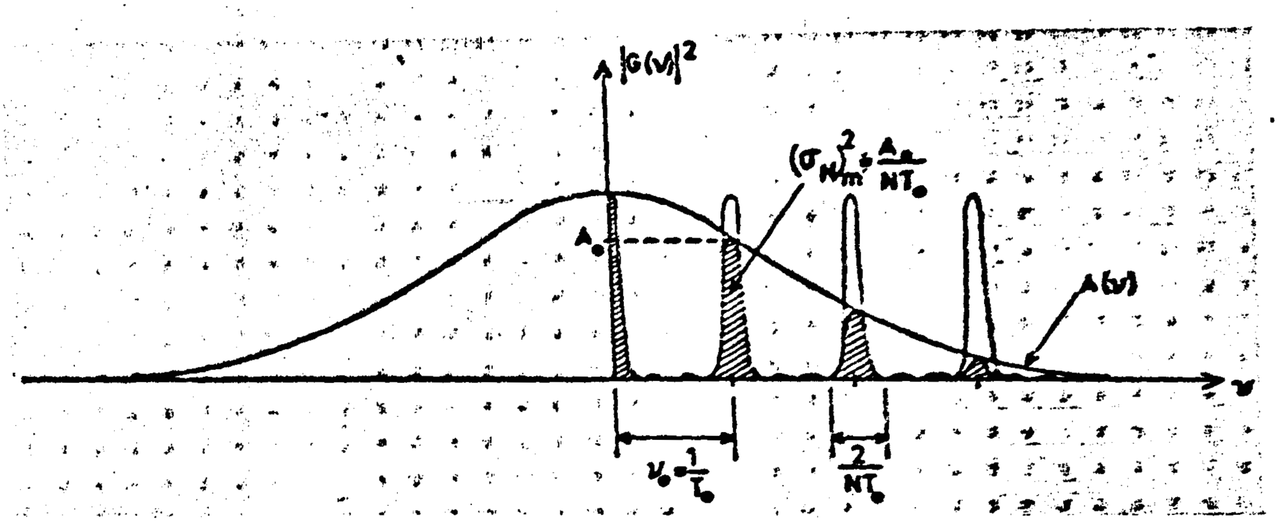

Figure 6. Noise energy after summation.

Y. W. Lee [16] we can find a demonstration of Equation (54) in an analogous form.

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IV. OPERATIONS IDENTICAL TO CONVOLUTION WITH A $g(t) = (2/T_0)\cos 2\pi\nu_0 t$.

A. Summation of N Periods and Convolution of the Result with a $g(t) = (2/NT_0)\cos 2\pi\nu_0 t$.

As far as the signal as well as the noise are concerned, there is an identity between:

- the convolution by $(2/NT_0)\cos 2\pi\nu_0 t$ of the recording taken over the interval $T + NT_0$ (with, let us recall, $\nu_0 = 1/T_0$);
- and the summation of N consecutive segments of width T_0, followed by the convolution of the result with $(2/T_0)\cos 2\pi\nu_0 t$.

In effect, we can easily establish the relationship

$$\begin{aligned} l(t) &= \frac{1}{NT_0}\int_{t_0}^{t_0+NT_0} f_0(\theta)\cos 2\pi\nu_0(t-\theta)\,d\theta \\ &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0}\left[\frac{1}{N}\sum_{k=0}^{N-1} f_0(\theta \right. \\ &\qquad \left. + kT_0)\right]\cos 2\pi\nu_0(t-\theta)\,d\theta, \end{aligned} \tag{65}$$

starting from

$$\begin{aligned} \int_{t_0}^{t_0+T_0} & f_0(\theta + kT_0)\cos 2\pi\nu_0(t_0-\theta)\,d\theta \\ &= \int_{t_0+kT_0}^{t_0+(k+1)T_0} f_0(\theta)\cos 2\pi\nu_0(t_0-\theta+kT_0)\,d\theta \\ &= \int_{t_0+kT_0}^{t_0+(k+1)T_0} f_0(\theta)\cos 2\pi\nu_0(t_0-\theta)\,d\theta. \end{aligned}$$

Furthermore, we verify that, for the noise, Equation (37) applied to $T + NT_0$,

$$\sigma_a^2 = \int_0^\infty A(\nu)\,|J_a(\nu)|^2_{NT_0}\,d\nu,$$

is identical to

$$\sigma_a^2 = \int_0^\infty A(\nu)\left(\frac{\sin \pi\nu N T_0}{N\sin \pi\nu T_0}\right)^2 |J_a(\nu)|^2_{T_0}\,d\nu.$$

On the other hand, as was seen in section II, we can write, w a very good approximation (Equations 44 and 47):

$$\sigma_s^2 \doteq \sigma^2 \doteq \frac{A_0}{NT_0}.$$

Now, this is precisely the same value

$$(\sigma_\Sigma^2)_m \doteq \frac{A_0}{NT_0}, \qquad (63)$$

that is obtained in the case of summation alone, when the filter which precedes it has a bandpass B less than or equal to $1/T_0$ and centered at $\nu_0 = 1/T_0$ (section IIIB). Thus when the filter is sufficiently selective, it is useless to carry out the convolution with $g(t) = (2/T_0) \cos 2\pi \nu_0 t$. In this form, the convolution therefore appears to be a process complementary to the summation, a process which allows us to obtain the minimum noise energy regardless of the selectivity of the electric filter, but which is superfluous when the latter is already sufficient.

B. Resolution into Fourier Series and Plot of the Term with Frequency ν_0.

We have seen in section IIB that the result l(t) of the convolution of $f_0(t)$ with $(2/NT_0) \cos 2\pi\nu_0 t$ on the interval $t_0 - NT_0/2 \leq \tau \leq t_0 + NT_0/2$ is sinusoidal with period $T_0 = 1/\nu_0$ whose coefficients are those, for the ν_0 term, of the infinite Fourier series which identically represents the recording of the signal and the noise in this interval. In practice, it is simpler to obtain l(t) and then to evaluate the two quantities:

$$\begin{aligned} a' &= \frac{2}{NT_0} \int_{t_0-NT_0/2}^{t_0+NT_0/2} f_0(\theta) \cos 2\pi \nu_0 \theta \, d\theta \\ b' &= \frac{2}{NT_0} \int_{t_0-NT_0/2}^{t_0+NT_0/2} f_0(\theta) \sin 2\pi \nu_0 \theta \, d\theta, \end{aligned} \qquad (66)$$

and to form

$$l_1(t) = a' \cos 2\pi \nu_0 t + b' \sin 2\pi \nu_0 t.$$

C. Study of the Sinusoidal Wave by Least Squares.

Until now, the processing of the recorded data has essentially been considered in terms of linearly smoothing the frequency spectrum. Because the signal is monochromatic, it is, in effect, normal to try to reduce the contribution of frequencies other than ν_0 and $-\nu_0$ in order to reduce the standard deviation of the noise. Since the ideal filter is one that does not pass frequencies other than ν_0 and $-\nu_0$ (for an infinite duration of observation) we have therefore considered the convolution by $\gamma(t) = (2/T) \cos 2\pi\nu_0 t$, a linear smoothing of gain:

$$G(\nu) = \frac{1}{T}[\delta(\nu + \nu_0) + \delta(\nu - \nu_0)]. \quad (35)$$

Another possible approach for improving the signal-to-noise ratio consists, this time in the domain of time, of studying the sinusoidal wave of frequency ν_0, which best lends itself, in the sense of least squares, to recording the signal and the noise. It is fairly remarkable that this second approach leads, in the case of a sinusoidal wave (as, moreover, in the case of a DC signal), to an identical result. In effect, let:

$$L(t) = A \cos 2\pi \nu_0 t + B \sin 2\pi \nu_0 t,$$

be a sinusoidal wave of frequency ν_0 such that:

$$E = \frac{1}{T}\int_{t_0-NT_0/2}^{t_0+NT_0/2} [f_0(t) - L(t)]^2 \, dt,$$

is minimum. This condition is realized by A and B such that:

$$\frac{\partial E}{\partial A} = \frac{\partial E}{\partial B} = 0,$$

that is [1, p. 67]:

$$A = \frac{2}{NT_0}\int_{t_0-NT_0/2}^{t_0+NT_0/2} f_0(\theta) \cos 2\pi \nu_0 \theta \, d\theta$$
$$B = \frac{2}{NT_0}\int_{t_0-NT_0/2}^{t_0+NT_0/2} f_0(\theta) \sin 2\pi \nu_0 \theta \, d\theta. \quad (67)$$

We have then, by comparing Equations (66) and (67);

$$A = a'_m$$
$$B = b'_m \quad (68)$$

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D. Summation of N Periods and Performing B or C on the Result.

For a recording of the duration T_o the operations defined in B and C are identical to convolution by $g(t) = (2/T_o) \cos 2\pi\nu_o t$. The conclusions of section A therefore apply immediately.

V. DETERMINATION OF THE AMPLITUDE AND THE PHASE OF THE SIGNAL. DISTRIBUTION LAWS.

Let:

$$l(t) = a' \cos 2\pi \nu_0 t + b' \sin 2\pi \nu_0 t = S' \cos (2\pi \nu_0 t + \varphi'),$$

be the result of convolution by:

$$g(t) = (2/NT_0) \cos 2\pi \nu_0 t,$$

of function:

$$f_0(t) = s_0(t) + x_0(t),$$

observed in the interval:

$$t_0 - (NT_0/2) \leq t \leq t_0 + (NT_0/2).$$

We have put:

(31

and we will assume in the following that the noise $x_0(t)$ is gaussia

The values of S' and ϕ' can be evaluated from two arbitrary points separated by $T_0/4$:

$$l(t_1) = S' \cos (2\pi \nu_0 t_1 + \varphi')$$
$$l\left(t_1 + \frac{T_0}{4}\right) = -S' \sin (2\pi \nu_0 t_0 + \varphi')$$

That is,

$$S' = \sqrt{l^2(t_1) + l^2\left(t_1 + \frac{T_0}{4}\right)}$$
$$\varphi' = \mathrm{Arc\,tg}\, \frac{-l(t_1 + T_0/4)}{l(t_1)} - 2\pi \nu_0 t_1.$$ (6

In particular, for $t_1 = kT_0$ we derive the Fourier coefficient

$$l(kT_0) = a'$$
$$l\left(kT_0 + \frac{T_0}{4}\right) = b',$$

with:

$$a' = \frac{2}{NT_0} \int_{t_0 - NT_0/2}^{t_0 + NT_0/2} f_0(\theta) \cos 2\pi\nu_0 \theta \, d\theta$$
$$b' = \frac{2}{NT_0} \int_{t_0 - NT_0/2}^{t_0 + NT_0/2} f_0(\theta) \sin 2\pi\nu_0 \theta \, d\theta.$$ (6

By putting:

$$t_1' = t_0 + \alpha,$$

and by replacing l(t) by its expression:

$$l(t)' = i(t) + y(t),$$

with:

$$i(t) = s_0(t) = S \cos(2\pi\nu_0 t + \varphi), \tag{33}$$

The Equation (69) can be written as:

$$S' = \sqrt{[s_0(t_0 + \alpha) + y(t_0 + \alpha)]^2 + \left[s_0\left(t_0 + \alpha + \frac{T_0}{4}\right) + y\left(t_0 + \alpha + \frac{T_0}{4}\right)\right]^2}$$
$$\varphi' = \text{Arc}\, tg \frac{-s_0(t_0 + \alpha + T_0/4) - y(t_0 + \alpha + T_0/4)}{s_0(t_0 + \alpha) + y(t_0 + \alpha)} - 2\pi\nu_0(t_0 + \alpha) \tag{70}$$

In particular, for $\alpha = \alpha_0$ such that:

$$t_1 = t_0 + \alpha_0 = kT_0,$$

we have:

$$S' = \sqrt{(a + \varepsilon_0)^2 + (b + \varepsilon_0)^2}, \tag{71}$$

$$tg\, \varphi' = -\frac{b + \varepsilon_0}{a + \varepsilon_0}, \tag{72}$$

by putting:

$$\varepsilon_0 = y(t_0 + \alpha_0) = \frac{2}{NT_0} \int_{t_0 - NT_0/2}^{t_0 + NT_0/2} x_0(\theta) \cos 2\pi\nu_0 \theta \, d\theta$$
$$\varepsilon_0 = y\left(t_0 + \alpha_0 + \frac{T_0}{4}\right) = \frac{2}{NT_0} \int_{t_0 - NT_0/2}^{t_0 + NT_0/2} x_0(\theta) \sin 2\pi\nu_0 \theta \, d\theta. \tag{73}$$

A. Statistical Properties of $y(t_0 + \alpha)$ and $y\left(t_0 + \alpha + \frac{T_0}{4}\right)$.

Just as we have seen in section IB, the quantities $y(t_0 + \alpha)$ and $y(t_0 + \alpha + T_0/4)$ are, just like $x_0(t)$, stationary random second order functions and with gaussian probability densities. They have an average value of zero and their standard deviations σ_α and $\sigma_{\alpha + T_0/4}$ are given by Equation (37). Their correlation coefficient is from Equation (A-12):

$$C_\alpha\left(\frac{T_0}{4}\right) \approx \cos 2\pi\nu_0\left(\frac{T_0}{4}\right) = 0.$$

These two quantities are therefore practically independent. In particular the same holds for the Fourier coefficients ε_a and ε_b defined by the Equation (73), and we will make use of this fact to stud[y] the statistical properties of S' and of ϕ'.

B. The Energy of the Signal and the Energy of the Noise.

From Equation (70) we deduce:

$$\overline{S'^2} = S^2 + \overline{y_a^2(t_0+\alpha)} + \overline{y_a^2(t_0+\alpha+T_0/4)},$$

or:

$$\overline{S'^2} = S^2 + \sigma_\alpha^2 + \sigma^2_{(\alpha+T_0/4)}. \tag{74}$$

The quantities σ_α^2 and $\sigma^2_{(\alpha+T_0/4)}$ are evaluated from Equation (37); in their sum, which we will designate by:

$$2\sigma^2 = \sigma_\alpha^2 + \sigma^2_{(\alpha+T_0/4)},$$

the cross term in $2\cos 4\pi\nu_0\alpha$ disappears, and we have exactly:

$$2\sigma^2 = 2\int_{-\infty}^{+\infty} A(\nu)\left(\frac{\sin \pi(\nu-\nu_0)NT_0}{\pi(\nu-\nu_0)NT_0}\right)^2 d\nu. \tag{75}$$

To the energy of the signal S^2 is added the energy of the nois[e] $2\sigma^2$ not too different from $2A_0/T$ (Equation 47).

Otherwise, and this time in an approximate form, we have:

$$\sigma_\alpha^2 \doteq \sigma^2_{(\alpha+T_0/4)} \doteq \sigma^2. \tag{76}$$

Similarly to the case of the summation, Equation (75) as well [as] the results relative to the case where there is sampling (section VI, B-b), have been initially obtained [32] by directly evaluating

$$2\sigma^2 = \overline{\varepsilon_a^2} + \overline{\varepsilon_b^2}$$

from the equation (73) by replacing:

$$\rho(\tau) = \overline{x_0(t).x_0(t-\tau)}.$$

by its value:

$$\rho(\tau) = \int_0^\infty A(\nu) \cos 2\pi \nu\tau \, d\nu.$$

Relative to the study of spectroscopy by Fourier transformation, Madame J. Connes [7, 8] has, in an approximate form which amounts to neglecting in $|J_a(\nu)|^2$ (defined by Equation 21) the cross term which contains α, extended the results of this section to a general case where the function h(t) which multiplies $f_0(t)$ is arbitrary (see also the work of J. Arsac [2, p. 326].

C. Distribution Law of S'.

We seek the probability density P(S') of:

$$S' = \sqrt{(a + \varepsilon_a)^2 + (b + \varepsilon_b)^2}, \tag{71}$$

with:

$$\bar{\varepsilon}_a = \bar{\varepsilon}_b = 0$$
$$\overline{\varepsilon_a \cdot \varepsilon_b} \doteq 0$$
$$\sigma^2_{\varepsilon_a} \doteq \sigma^2_{\varepsilon_b} \doteq \sigma^2,$$

and the probability densities:

$$P(\varepsilon_a) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\varepsilon_a^2/2\sigma^2\right]$$
$$P(\varepsilon_b) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\varepsilon_b^2/2\sigma^2\right].$$

For a different problem, that of determining the distribution law of the envelope of a carrier signal onto which a gaussian noise is superposed, the probability density of an analogous quantity S' but with b = 0, has been equated by several authors (for example, see references [10 and 11]). Here we take up these calculations by generalizing to the case where b ≠ 0.

Let:

$$x = a + \varepsilon_a$$
$$y = -b - \varepsilon_b.$$

The probability of having both x and y, respectively equal to two given values or a probability density at a point M in the plane (Figure 7) is:

$$P_M = P(x).P(y) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{(x-a)^2+(y+b)^2}{2\sigma^2}\right]$$

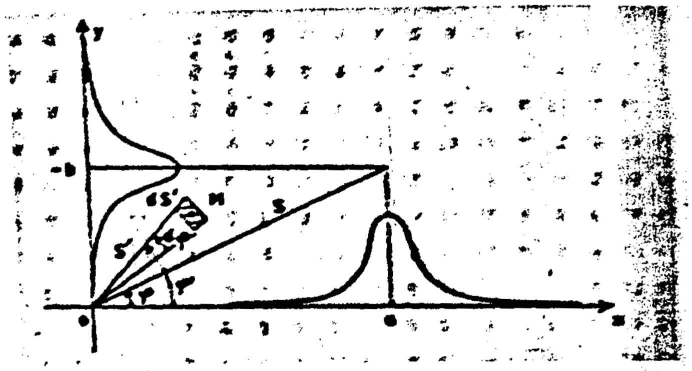

Figure 7. Geometric representation of the composition of the two gaussian laws.

Replacing a and b by their values $a = S\cos\phi$ and $b = -S\sin\phi$ and expressing S' and ϕ' in polar coordinates, we obtain:

$$P_M = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{S^2+S'^2-2SS'\cos(\varphi'-\varphi)}{2\sigma^2}\right]. \quad (77)$$

As the probability of finding points in the surface element $ds = S'\,dS'\,d\phi'$ is:

$$P_{ds} = P_M S'\,dS'\,d\varphi',$$

we have:

$$P(S')\,dS' = dS' \int_{\varphi=0}^{2\pi} P_M . S'\,d\varphi', \quad (78$$

that is:

$$P(S') = \frac{S'}{2\pi\sigma^2}\exp\left[-\frac{S^2+S'^2}{2\sigma^2}\right]\int_0^{2\pi} \exp\left[\frac{SS'\cos(\varphi'-\varphi)}{\sigma^2}\right] d\varphi'.$$

This integral is easily expressed as a modified Bessel functi of the first kind $I_0(z)$ and we finally have:

$$\boxed{P(S') = \frac{S'}{\sigma^2} I_0\left(\frac{SS'}{\sigma^2}\right)\exp\left[-\frac{S^2+S'^2}{2\sigma^2}\right]}. \quad ($$

Figure 8, from reference [11] represents P(S') as a function of S'/σ for different values of the parameter:

$$\eta = \frac{S}{\sigma},$$

the ratio of the amplitude of the sinusoidal signal to the standard deviation of ε_a and ε_b. Let us recall that σ is $\sqrt{BNT}$ times smaller than the standard deviation of the noise in the recording from which we have evaluated the Fourier coefficients (Equation 49).

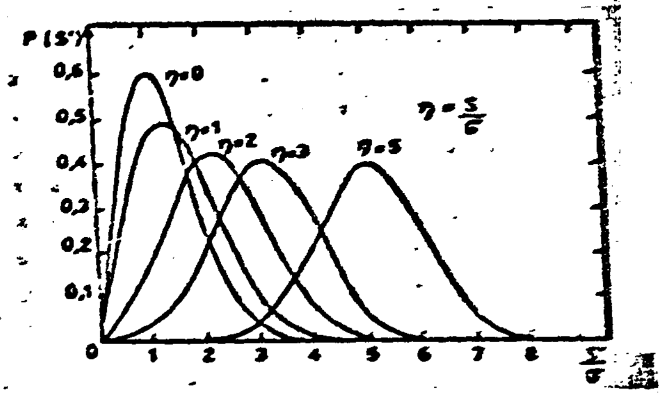

Figure 8. Probability density of the amplitude (S is the true value and σ is the standard deviation of the noise after convolution with a sinusoidal wave.

Average Value and Standard Deviation.

We have:

$$\overline{S'} = \int_{-\infty}^{+\infty} S' \, P(S') \, dS' \qquad (80)$$

$$\sigma_{S'}^2 = \overline{(S' - \overline{S'})^2} = \int_{-\infty}^{+\infty} (S' - \overline{S'})^2 \, P(S') \, dS'.$$

These quantities can be expressed simply as a function of S and of σ in the two cases $\frac{S}{\sigma} \ll 1$ and $\frac{S}{\sigma} \gg 1$ [11].

In the first case $\frac{S}{\sigma} \ll 1$, Equations (80) become:

$$\left\{\begin{aligned} \overline{S'} &\doteq \sqrt{\frac{\pi}{2}}\,\sigma\left(1 + \frac{S^2}{4\sigma^2}\right) \\ \sigma_{S'}^2 &\doteq \left(2 - \frac{\pi}{2}\right)\sigma^2\left(1 + \frac{S^2}{4\sigma^2}\right). \end{aligned}\right. \qquad (81)$$

When the ratio $\eta = S/\sigma$ is small, added to statistical error characterized by σ_s is a systematic indeterminacy in the value of S as the average value of S' is a function of σ (we will find an analogous conclusion deduced from direct evaluation, proceeding from S' and σ_s^2, in reference [7]). This indeterminacy could not be eliminated except perhaps by successive approximations (it is meanwhile possible to obtain an approximate value of S at the price of an independent evaluation of σ either by calculation or by statistical study of the recording of pure noise x(t)). This shows the advantage that there is in not carrying out a non-linear operation which is the evaluation of S' by Equation (71) which once obtained the best possible signal-to-noise ratio.

Thus, when we have a recording the length $T + NT_o$ at our disposal it is recommended:

- that first the summation of the N periods be carried out, then the amplitudes and the phase of the sinusoidal wave of period T_o be evaluated from the result;

- rather than evaluating these quantities independently for each of the elemental periods, and then carrying out the arithmetic average of the N separate determinations.

2) $\frac{S}{\sigma} \gg 1.$

We have:

$$\overline{S'} \doteqdot S\left(1 + \frac{1}{2\frac{S^2}{\sigma^2}}\right). \tag{82}$$

$$\sigma_{S'}^2 \doteqdot \sigma^2$$

The average value of S' this time is very little different from S. As to the standard deviation σ_s, it is practically equal to that of ε_a and ε_b, that is, the standard deviation of the result y(t) of the convolution of the noise with $g(t) = (2/NT_o) \cos 2\pi\nu_o t$.

Furthermore, we see from Figure 8 that P(S') is very rapidly approaching a gaussian distribution centered on S. When the quantity $\eta = S/\sigma$ is large, the signal-to-noise ratio of the amplitude of the signal to the standard deviation relative to this amplitude is very close to:

$$\boxed{\frac{\overline{S'}}{\sigma_{N}} \doteq \frac{S}{\sigma}} \tag{83}$$

which, with Equation (47), is:

$$\sigma \doteq \sqrt{\frac{A_0}{T}} \tag{84}$$

D. <u>Distribution Law of ϕ'.</u>

The phase ϕ' is defined by:

$$tg\varphi' = -\frac{b + \varepsilon_b}{a + \varepsilon_a}, \tag{72}$$

that is:

$$tg\varphi' = \frac{y}{x},$$

The respective signs of x and y indicate in which quadrant the angle ϕ' happens to fall between 0 and 2π. Its probability density $P(\phi')$ is evaluated in the same manner as in the above section by writing that the probability that the phase ϕ' should be within the element $d\phi'$ is:

$$P(\varphi')\, d\varphi' = d\varphi' \int_{S'=0}^{\infty} P_M S'\, dS', \tag{85}$$

P_M is given by Equation (77), which leads to:

$$P(\varphi') = \frac{1}{2\pi\sigma^2} \int_0^{\infty} \exp\left[\frac{-S^2 + S'^2 - 2SS' \cos(\varphi' - \varphi)}{2\sigma^2}\right] S'\, dS'.$$

After transformation, it gives:

$$\boxed{\begin{aligned} P(\varphi') = \exp\left[-\frac{S^2 \sin^2(\varphi'-\varphi)}{2\sigma^2}\right] \times \\ \left\{\frac{S\cos(\varphi'-\varphi)}{\sigma\sqrt{2\pi}}\left[\frac{1}{2} \pm K\left(\frac{S\cos(\varphi'-\varphi)}{\sigma}\right)\right] \right. \\ \left. + \frac{1}{2\pi}\exp\left[\frac{-S^2\cos^2(\varphi'-\varphi)}{2\sigma^2}\right]\right\} \end{aligned}} \qquad (86)$$

with:

$$K(u) = \frac{1}{\sqrt{2\pi}}\int_0^u \exp\left[-\frac{z^2}{2}\right] dz \qquad (87)$$

which is a function related to the error function and which is found tabulated, for example, in reference [9]. The sign in front of $K\,[S\cos(\varphi'-\varphi)/\sigma]$ is positive whenever $\cos(\phi' - \phi)$ is positive, and negative in the other case.

In Figure 9 we have shown $P(\varphi'-\varphi)$ as a function of $(\phi' - \phi)$ for various values of the parameter: $\eta = \frac{S}{\sigma}$

When S/σ is large, this function approaches:

$$P(\varphi') = \frac{S}{\sigma\sqrt{2\pi}}\exp\left[\frac{-S^2\sin^2(\varphi'-\varphi)}{2\sigma^2}\right]. \qquad (88)$$

which is very similar to a gaussian distribution, centered at $\phi' = \phi$

$$\overline{\varphi'} = \varphi.$$

and with a standard deviation:

$$\boxed{\sigma_{\varphi'} = \frac{S}{\sigma}} \qquad (89)$$

where $\sigma_{\phi'}$ is expressed in radians.

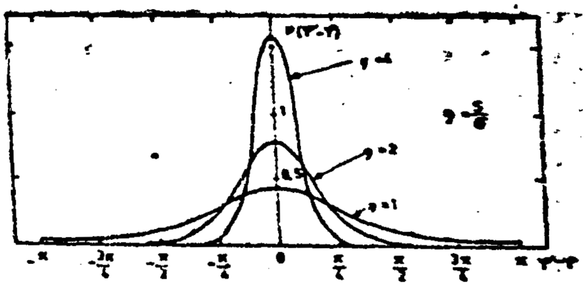

Figure 9. Probability density of the phase.

VI. PROPERTIES AND FUNCTIONS OF THE ELECTRIC FILTER.

The electric filter which precedes the convolution (block 2 in Figure 1) serves a double purpose.

First of all, it is directly involved as the linear filter that allowed us to improve the signal-to-noise ratio, and in this regard it could be considered a priori as a complement or as a possible substitute to convolution.

Then, and this is above all, what gives importance to this filter, it allows us, at the price of a slight reduction in the final signal-to-noise ratio, to reduce to a minimum the quantity of data to be used in the calculation. This is done in two complementary forms: sampling and quantization. On the one hand, in effect, the noise after filtering, has a statistical memory related to its correlation function [5], and the result then is a certain redundance of data which allows us to use only discrete values obtained by sampling. On the other hand, because the signal is contaminated by noise, it is sufficient to carry out the measurements with a precision as low as the signal-to-noise ratio is small.

These three aspects of the electric filter are considered in what follows.

A. Smoothing

The electric filters are a particular class of linear filters characterized by the fact that the percussion response $g_E(t)$ is zero for t negative. Corresponding to a periodic or constant function $f_o(t)$ applied at the instant t = 0, is a response l(t) which evolves from a transient range of periodicity toward a permanent range $l_p(t)$ such that:

$$l_P(t) = f_o(t) * g_E(t).$$

a. Transient range

The transient range, defined as the time at the beginning of which the amplitude of the response function is but a fraction of that corresponding to the permanent range, is of the order of the inverse of the bandpass of the filter. For example, in the case of a narrow band filter, Küpfmuller, quoted among others in reference [25], has shown that the time Δt required for the current to pass from the 10th to the 9th tenth of its final value is

$$\Delta t = \frac{0,87}{B},$$

where B is defined as the interval between the frequencies for which the attenuation is $1/\sqrt{e}$ (where e is the exponential). Similarly, Arsac [2, p. 239] has shown by applying Bernstein's theorem, that in case of a low pass filter and a signal zero for $t < 0$ and constant $t > 0$, the time Δt necessary for the response function to reach its maximum is such that:

$$\Delta t \geqslant \frac{1}{\pi B},$$

where B is the maximum frequency transmitted by the lowpass filter.

b. Permanent range

As far as the signal is concerned, we have

$$i_p(t) = s_0(t) * g_E(t)$$
$$I_p(\nu) = S_0(\nu) . G_E(\nu),$$

with

$$S_0(\nu) = \frac{S}{2} e^{i\varphi} \delta(\nu + \nu_0) + \frac{S}{2} e^{-i\varphi} \delta(\nu - \nu_0), \qquad (55$$

which leads to

$$I_p(\nu) = \frac{S}{2} e^{i\varphi} G_E(-\nu_0) \delta(\nu + \nu_0) + \frac{S}{2} e^{-i\varphi} G_E(\nu_0) \delta(\nu - \nu_0).$$

The effect of the filter is to modify the phase of the signal and to multiply its amplitude by:

$$|G_E(-\nu_0)| = |G_E(\nu_0)|$$

As far as the noise is concerned, its standard deviation is given by (section IA):

$$\sigma^2 = \int_0^\infty A(\nu)\, d\nu,$$

with:

$$A(\nu) = A_i(\nu) \cdot |G_E(\nu)|^2$$

where $A_1(\nu)$ is the spectral energy density of the noise before smoothing. We will assume in the following that $A_1(\nu)$ is constant (white noise) with the value A_1, which leads to:

$$A(\nu) = A_1\, |G_E(\nu)|^2$$

c. Electric smoothing alone

We could think of using exclusively an electric filter of very narrow bandpass centered on the frequency ν_0. We would then evaluate graphically the amplitude and phase of a period of the sinusoidal image on a portion of the recording corresponding to the permanent range (or considered as such) and from which we could deduce the amplitude and the phase of $s_0(t)$ from the supposedly known characteristics of the filter.

For such a method to be possible, the filter should not be too selective in order that the duration of the transient range not exceed that of the window of observation, which leads to the condition

$$B > \frac{1}{T}$$

For this to be efficient, that is, for this to allow us to obtain all available data without necessitating a supplementary smoothing operation, the duration of the transient range should be just a little smaller than T. Lacking this condition, in effect, of all the recordings in the permanent range only an interval equal to one period of this signal would be used.

When, on the other hand, this two-fold condition is fulfilled, that is, when the bandpass of the filter is of the order of 1/T, then the use of an electric filter by itself leads to a signal-to-noise ratio close to one given by the convolution with $g(t) = (2/T)\cos 2\pi\nu_0 t$:

$$\frac{s}{\sigma} = \frac{s}{\sqrt{\int_0^\infty A_i(\nu)|G_E(\nu)|^2 \, d\nu}} \simeq \frac{s}{\sqrt{A_i B}} \simeq \frac{s}{\sqrt{A_i/T}}.$$

In effect, this solution is hardly practicable, because it requires:

- having observation windows of the same duration T;

- performing, when T is very large, a very selective smoothing and then we come face to face with technical difficulties;

- modifying the tuning of the filter as a function of ν_0.

d. Electric and mathematical smoothing

The convolution with $g(t) = (2/T)\cos 2\pi\nu_0 t$ is specifically applicable as we have seen, to the case when the signal is sinusoidal. That is when an electric filter precedes the recorder, the convolution can be carried out only on the part T' of the recording corresponding to the permanent range

$$T' = T - \Delta t.$$

With this generally negligible correction, the results of section II are immediately applicable. We recall in particular that the standard deviation of the noise after convolution with $g(t) = (2/T')\cos$ has the value (from Equation 47):

$$\sigma \doteq \sqrt{\frac{A_0}{T'}}$$

where A_0 is the spectral energy density of the noise in the record at the frequency ν_0. With the notation of section VI Ab, we have

$$A_0 = A_i\,|G_E(\nu_0)|^2$$

which leads to:

$$\sigma \doteq A_i\,|G_E(\nu_0)|\sqrt{\frac{A_i}{T'}}$$

Similarly, the electric filter has the effect of multiplying the amplitude of the signal by $|G_2(\nu_0)|$. The result is that the final signal-to-noise ratio is practically independent of the characteristics of this filter.

B. Sampling Followed by Convolution with a Sinusoidal Wave.

When the convolution with a sinusoidal wave (or one of the operations which lead to an identical result) is carried out in digital form, we sample the observed function by seeking to give to the sampling interval τ the largest possible value τ_o.

In a completely general fashion, given a function consisting of the sum of a signal and a noise, a function to which we apply a given mathematical operator, the problem of choosing the maximum sampling interval is related to the following two-fold condition:

- as far as the signal is concerned, it should not have an appreciable loss of information (in other words, the sampling should not noticeably affect the form of the final result);

- as far as the noise is concerned, the standard deviation after sampling and the application of the mathematical operator under consideration should not exceed, by more than a given percentage considered acceptable, the standard deviation that we would have in the absence of sampling.

These conditions are independent, and lead to two different values $(\tau_o)_S$ and $(\tau_o)_B$, respectively, where the optimum interval τ_o is evidently the smaller of these two quantities. Therefore, on the condition that these calculations are present in a sufficiently simple form, the evaluation of τ_o can be carried out in a rigorous manner: as the function is assumed to be sampled with an interval τ, we apply to such a series of discrete points the mathematical operator under consideration, and we evaluate the result as far as it concerns the signal as well as the noise. We can therefore judge the degree of degradation in the quality of this result as a function of τ, and thus decide on the highest acceptable value of τ, i.e., τ_o.

All this obviously assumes that the statistical properties of the noise are known, and that we have a minimum of data on the analytical form of the signal. In the case where this last condition is not realized, it is possible, in general, to evaluate $(\tau_c)_S$ approximately by a different approach: that of the sampling theorem of Shannon [28]. In effect, we know that when a signal $s_c(t)$ has a Fourier transform strictly limited to an interval B, an infinite series of equidistant discrete values at intervals less than or equal to 1/2B is sufficient to define $s_o(t)$. We have:

$$s_0(t) = \sum_{k=-\infty}^{+\infty} s_0\left(\frac{k}{2B}\right)\frac{\sin 2\pi B(t - k/2B)}{2\pi B(t - k/2B)},$$

as the limiting case of the representation of the function by a sum of translated values [3]. In fact, the duration of observation cannot be infinite, and the spectrum of the signal is generally not finite. The quantity 1/2B, meanwhile, represents a first approximation, as far as the signal and only the signal is concerned, of the upper limit of the sampling interval.

In the present case, that of the convolution, with a sinusoidal wave, of the recording over a finite duration T, made up of the sum of a sinusoidal wave of the same periodicity and a stationary noise the rigorous evaluation of $(\tau_o)_S$ and of $(\tau_o)_B$ does not present any difficulty. For that of the signal $s_o(t)$, it is sufficient, as we will see below, that the sampling interval by a submultiple of $T = NT_o$ different from an integral number of half periods, for the result of the convolution to be identical to $s_o(t)$. As to the evaluation of $(\tau_o)_B$, for the noise, it follows immediately from the results established in section I: $(\tau_o)_B$ is a function of both $A(\nu) = A_0|G_B(\nu)|^2$, hence the energy response in terms of frequency of the filter which precedes the recording, and of $G(\nu)$, the spectrum of the function $g(t)$ with which it is convoluted.

Therefore, let:

$$f(t) = \sum_{k=0}^{n-2} f_0(t_0 + k\tau)\,\delta[t - (t_0 + k\tau)]\qquad (90)$$

be the result of sampling $f_0(t)$, taken on the interval $t_0 \leq t \leq t_0 + NT_0$ with the sampling step being $\tau = NT_0/(n-1)$ (section I-C). The highest term in the series is taken to be (n - 2) instead of (n - 1) for the reasons indicated below.

This distribution f(t) is convoluted with:

$$g(t) = \frac{2}{n-1}\cos 2\pi\nu_0 t\ ;$$

The result is, as we can immediately verify, a sinusoidal wave of the form:

$$l(t) = a'\cos 2\pi\nu_0 t + b'\sin 2\pi\nu_0 t,\qquad (91)$$

with

$$\begin{aligned} a' &= \frac{2}{n-1}\sum_{k=0}^{n-2} f(t_0 + k\tau)\cos 2\pi\nu_0(t_0 + k\tau) \\ b' &= \frac{2}{n-1}\sum_{k=0}^{n-2} f(t_0 + k\tau)\sin 2\pi\nu_0(t_0 + k\tau) \end{aligned}\qquad (92)$$

a. Effect on the signal.

When the highest term in the series is (n - 2), the result i(t) of the sampling and of the convolution is a sinusoidal wave identical to the sinusoidal object $s_0(t)$, whatever the number of sampled points, with the only condition that τ be a submultiple of $T = NT_0$, different from an integral number of half-periods. We can show this simply by replacing, in the expression of the Fourier transform of i(t):

$$I(\nu) = [S_0(\nu) * H(\nu)\,e^{2\pi i\nu t_0}].G(\nu),\qquad (14)$$

the functions $S_0(\nu)$ and $G(\nu)$ by their values (Equations 35 and 55) and by:

$$H(\nu) = \frac{\sin \pi\nu(n-1)\tau}{\sin \pi\nu\tau}\,e^{i\pi\nu(n-2)\tau},\qquad (93)$$

obtained from Equation (26) by replacing n by n-1. We find that under the indicated conditions $I(\nu)$ is equal to $S_o(\nu)$.

Applied to the signal $s_o(t)$, the equations (92) therefore provide its Fourier coefficients a and b (this result is given in reference [1, p. 729] in a more general form, but with a slight error As a result, we have:

$$\begin{aligned} a' &= a + \varepsilon_a' \\ b' &= b + \varepsilon_b' \end{aligned} \qquad (94)$$

where ε_a' and ε_b' are, same as ε_a and ε_b given by the Equations (73), the Fourier coefficients of the random sinusoidal wave which is superimposed on the signal.

b. Effect on the noise.

The bibliographical references are the same as those of section V-B, and furthermore the conclusions of sections I, II and V are directly applicable. In particular, the standard deviation σ_α at a point located at a distance $t_o + \alpha$ from the origin is given by the Equations (21), where $H(\nu)$ is replaced by the expression (93).

Putting

$$\begin{aligned} \pi\nu\tau &= \omega \\ \pi\nu_0\tau &= \omega_0 \\ n - 1 &= n' \end{aligned}$$

we obtain

$$\sigma_\alpha^2 = \int_0^\infty A(\nu) \left[\left(\frac{\sin n'(\omega + \omega_0)}{n' \sin(\omega + \omega_0)} \right)^2 + \left(\frac{\sin n'(\omega - \omega_0)}{n' \sin(\omega - \omega_0)} \right)^2 + 2 \cos 4\pi\nu_0 \alpha \frac{\sin n'(\omega + \omega_0) . \sin n'(\omega - \omega_0)}{n' \sin(\omega + \omega_0) . n' \sin(\omega - \omega_0)} \right] d\nu. \qquad (9$$

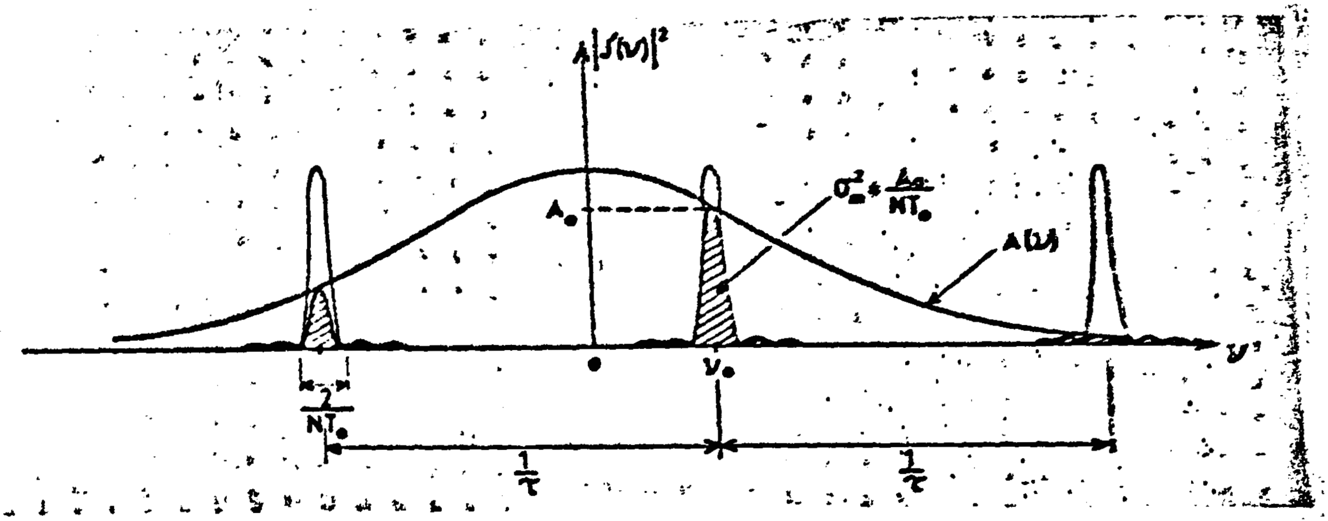

Figure 10. Noise energy after sampling at interval τ, followed by convolution with a sinusoidal wave.

When τ approaches zero, we again have Equation (37). Same as in section II, the contribution of the cross-term is completely negligible, such that σ_α is practically independent of α and is equal to σ such that:

$$\sigma^2 = \int_0^\infty A(\nu)\left[\left(\frac{\sin\pi(\nu+\nu_0)(n-1)\tau}{(n-1)\sin\pi(\nu+\nu_0)\tau}\right)^2 + \left(\frac{\sin\pi(\nu-\nu_0)(n-1)\tau}{(n-1)\sin\pi(\nu-\nu_0)\tau}\right)^2\right]d\nu,$$

which can be written, by putting $A(-\nu) = A(\nu)$, as

$$\boxed{\sigma_\alpha^2 \doteq \sigma^2 = \int_{-\infty}^{+\infty} A(\nu)\left(\frac{\sin\pi(\nu-\nu_0)(n-1)\tau}{(n-1)\sin\pi(\nu-\nu_0)\tau}\right)^2 d\nu} \quad (96)$$

with:

$$(n-1)\tau = NT_0. \quad (97)$$

The function $|J'(\nu)|^2$ which multiplies $A(\nu)$ is shown in Figure 10. It is periodic, and includes a series of main lobes:.

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- centered on the absissae:

$$\nu_{\pm p} = \nu_0 \pm p/\tau,$$

- of base $2/NT_o$
- of height 1
- of area, including the related secondary lobes, equal to $1/NT$

The noise energy σ^2 is therefore the sum of the energy σ_m^2 corresponding to the central lobe, and to the energy $\Delta\sigma^2$ corresponding to the areas delineated by the lobes at absissae $\nu_0 \pm p/\tau$ in the curve $A(\nu)$, the product of the spectral energy density A_1 of the noise before smoothing by the energy response function in terms of frequency of the electric filter, $|G_E(\nu)|^2$. If B is the maximum frequency transmitted by this filter, it is sufficient then that the sampling interval be such that:

$$\frac{1}{\tau} \geqslant B + \nu_0 + \frac{\nu_0}{N} \tag{98}$$

for the σ^2 to be equal to σ_m^2, the minimum noise energy that we could have in the absence of sampling.

In the general case where $A(\nu)$ does not vanish beyond a certain value, we could evaluate, either directly or in graphical form from Figure 10, the loss of information represented by the ratio $\Delta\sigma^2/\sigma_m^2$ as a function of τ. Such an evaluation is carried out in section V in the case of smoothing by integrating over a finite range, which seems to be preferable to electric smoothing.

c. Remark: case of a DC signal.

As the function g(t) is, in the case of a DC signal, replaced $g(t) = \frac{1}{n}$, Equation (95) becomes:

$$\sigma^2 = \int_0^\infty A(\nu)\left(\frac{\sin \pi\nu n\tau}{n \sin \pi\nu\tau}\right)^2 d\nu$$

a result that was already obtained elsewhere relative to the study summation (Equation 60).

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C. Quantization of the Amplitude.

Let l_k be the value, sampled at the abscissa $t_o + k\tau$ of $l(t) = i(t) + y(t)$, the output function of the electric filter. By quantization of the amplitude we mean the numerical representation of l_k, that is, the operation which establishes a correspondence between l_k and an integer L_k such that

$$L_k - \frac{1}{2} < \frac{l_k}{q} < L_k + \frac{1}{2} \tag{99}$$

where q is the width of the quantization step (Figure 11).

This question has been treated by several authors [15, 34], and in particular by B. Widrow [35], to whom the following results are attributed.

The nonlinear quantization operator has the effect of introducing an additional noise:

$$Y_k = L_k - l_k. \tag{100}$$

which in general is correlated with l_k.

Meanwhile, when the quantization step q is sufficiently small relative to the range of variation of the noise y_k, the degree of correlation is practically zero, and everything proceeds as if the quantization operator behaved as an additional source of noise independent of Y_k. Under these conditions, the probability density of Y_k is uniform over the interval - q/2 to + q/2 and is zero beyond that:

$$\begin{aligned} P(Y_k) &= \frac{1}{q} \text{ for } -q/2 \leq Y_k \leq +q/2 \\ &= 0 \text{ for } |Y_k| \leq q/2. \end{aligned} \tag{101}$$

From this we deduce the average value and the standard deviation of Y_k with the use of the definitions:

$$\overline{Y}_k = \int_{-\infty}^{+\infty} Y_k P(Y_k)\, dY_k$$

$$\sigma^2_{Y_k} = \int_{-\infty}^{+\infty} (Y_k - \overline{Y}_k)^2 P(Y_k)\, dY_k,$$

which give:

$$\overline{Y}_k = 0$$

$$\sigma_{Y_k} = \sigma_q = \frac{q}{\sqrt{12}}. \qquad (102)$$

We put $\sigma_{yk} = \sigma_q$ to emphasize that it consists of a standard deviation due to quantization.

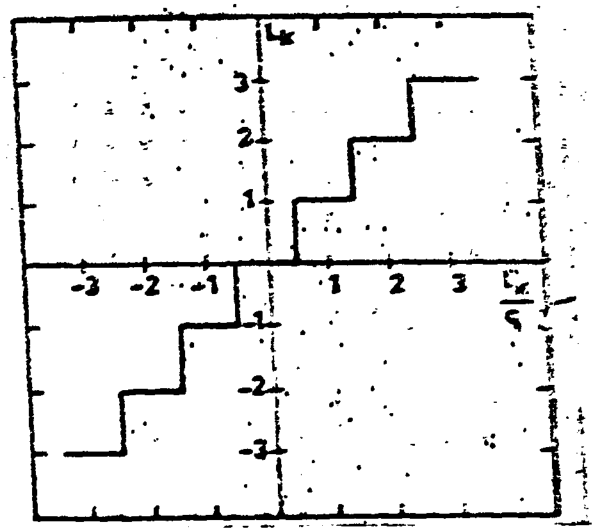

Figure 11. The amplitude quantization principle.

In the case where y(t) is gaussian, all this is applied with a very good approximation when the quantization step is less than the standard deviation σ_o of y(t), that is,

$$q \leqslant \sigma_o.$$

After quantization, the total noise that affects s_k is:

$$y_k' = y_k + Y_k, \qquad (103)$$

with a standard deviation of

$$\sigma_1' = \sqrt{\sigma_0^2 + \sigma_q^2}$$

which approaches σ_o as the ratio q/σ_o becomes smaller. Furthermore, the probability density $P(y_k')$ is equal to the convolution product of $P(y_k)$ and $P(Y_k)$.

Quantization and Summation.

From the preceding we can easily deduce the statistical properties of the average:

$$L_N = \frac{1}{N} \sum_{p=0}^{N-1} L_p,$$

where L_p is the quantified value of $l_p = i_p + y_p$ at the abscissa $t_0 + k\tau + p(n-1)\tau$, with k constant and $(n - 1)\tau = T_o$. This is the type of average that we take when we are performing the convolutio of l(t) with a g(t) = $(2/NT_0)\cos 2\pi\nu_0 t$ de $l(t)$ on an interval NT_0 (sections II and III). We recall that i_p is then a constant which we will designate by i.

Onto the signal i and the noise $y_N = \sum_{p=0}^{N-1} y_p$ we superimpose the noise of quantization:

$$Y_N = \sum_{p=0}^{N-1} Y_p \tag{104}$$

As the random variables Y_p are independent, the probability density $P(Y_N)$ of Y_N is equal to the result of the convolution product of $P(Y_p)$ with itself, carried out (N-1) times. In other words, the Fourier transform, or the characteristic function $P(Y_N)$:

$$\varphi(u)_N = \int_{-\infty}^{+\infty} P(Y_N)\, e^{juY_N}\, dY_N,$$

is equal to the Nth power of the characteristic function $\phi(u)$ of $P(Y_p)$:

$$\varphi(u)_N = [\varphi(u)]^N = \left(\frac{\sin\frac{qu}{2}}{qu/2}\right)^N.$$

We see in Figure 12 where $P(Y_N)$ is shown for different values of N, that this probability density very rapidly approaches a gaussian curve as N increases. This can be shown quantitatively [33] by comparing $P(Y_N)$ to the gaussian distribution having the same average value and the same standard deviation, by means of a development by Gram-Charlier [1].

Under these conditions, we could immediately evaluate the effect of the width of the quantization step q on the final signal-to-noise ratio. The total noise, after quantization and summation, is the sum of two independent gaussian noises: the physical noise whose energy is given by Equation (60) and (62), has the value:

$$\sigma_N^2 = \int_0^\infty \Delta(\nu) \left(\frac{\sin \pi\nu NT_0}{N \sin \pi\nu T_0}\right)^2 d\nu \simeq \frac{\sigma_0^2}{N}.$$

and the quantization noise, with the energy equal to:

$$(\sigma_e)^2_N = \frac{q^2}{12N}.$$

This resultant noise is also gaussian with a standard deviation of:

$$\sigma'_N = \sqrt{\sigma_N^2 + (\sigma_e)^2_N} \doteq \sigma_N \sqrt{1 + \frac{1}{12}\frac{q^2}{\sigma_0^2}}$$

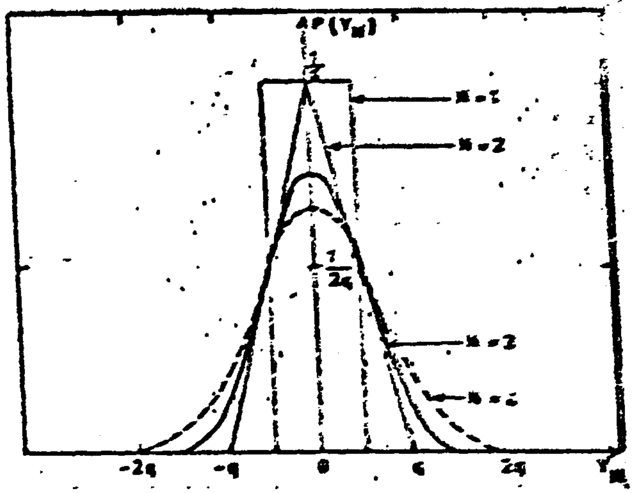

Figure 12. Probability density of the noise of quantization, as a function of the number of additional samples.

The ratio σ'_N/σ_N is practically independent of N, and we will designate it by σ'/σ, σ' being the standard deviation that we would have in the absence of quantization:

$$\frac{\sigma'}{\sigma} \simeq \sqrt{1 + \frac{1}{12}\frac{q^2}{\sigma_0^2}} \qquad (105)$$

This quantity is shown in Figure 13 as a function of σ_0/q, the ratio of the standard deviation of the noise in the recording and the width q of the quantization step. We see that there is little to be gained by taking q smaller than σ_0. For $q = \sigma_0$, the final standard deviation σ' is greater by no more than 4% than it would be in the absence of quantization.

D. Pre-Smoothing and Quantity of Data to be Used.

The essential role of the filter that precedes recording is, as we have seen, to reduce to a minimum the quantity of data to be used in the calculations. This reduction in the quantity of useful data shows up in two complementary forms: the sampling and the quantization. Both of these operations have the effect of introducing additional noise. As far as the sampling is concerned, the interval τ, corresponding to a given value of such an excess noise, is very nearly inversely proportional to the bandpass B of the filter. On the other hand, as far as the quantization is concerned, the step q is noticeably proportional to $\sqrt{B}$. On balance, these two operations show that there is an advantage in using a filter with as narrow a bandpass B as possible, but not too close to the inverse of the observation duration (by reason of the transient range).

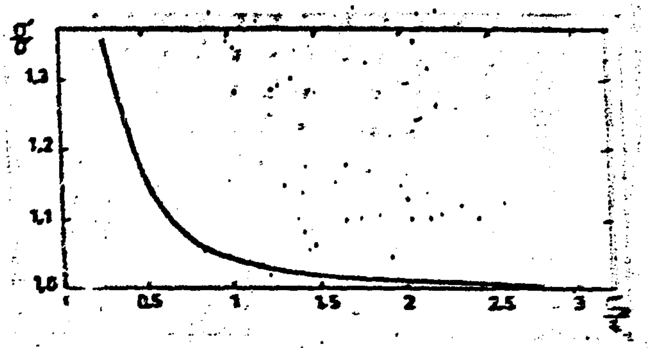

Figure 13. Relative increase in the standard deviation of the noise due to quantization.

On the other hand, from the fact that the sampling noise and that of quantization are independent, the standard deviation of their sum is equal to the square root of the sum of their energies. It therefore is useless to try and reduce one noticeably more than the other.

Given the characteristics of the filter, and the acceptable excess noise energy having been defined, the values of τ and q can then be deduced. Taking into account the initial signal-to-noise ratio and the duration of observation, we can therefore evaluate the quantity of data to be recorded.

VII. SMOOTHING BY INTEGRATING OVER A LIMITED RANGE.

The use of an electric filter as a means of pre-smoothing represents a two-fold inconvenience: "endowed" with a time constant, such a filter has an effect on the characteristics by dephasing the signal and by having a non-limited temporary memory that can be annoying in case of external noise.

There exists at least one form of smoothing that at the same time performs the sampling function, and which does not have these defects. It consists of transforming the continuous function $f_c(t)$ into a function $l(t)$ formed of discrete equidistant points at intervals τ such that:

$$l(t_0 + k\tau) = \frac{1}{\tau}\int_{t_0+k\tau-\tau/2}^{t_0+k\tau+\tau/2} f_0(\theta)\,d\theta. \quad (106$$

The point $l(t_0 + k\tau)$ is obtained as the result of integrating $f_0(t)$ in the interval τ about the point $t_0 + k\tau$ (Figure 14).

In practice this situation can be realized, for example, by means of an analog-digital converter based on a principle of converting a voltage into a series of pulses which is proportional to it. By accumulating these pulses in a counter, during the time interval τ, we obtain a number proportional to $l(t_0 + k\tau)$. This technique h for example, been systematically employed by the National Radio Astronomy Conservatory at Green Bank.

The transformation of $f_0(t)$ into $l(t)$, defined above, in fact represents the combination of the following two operations:

$$l_0(t) = \frac{1}{\tau}\int_{t-\tau/2}^{t+\tau/2} f_0(\theta)\,d\theta \quad (1$$

$$l(t) = \sum_{k=0}^{n-1} l_0(t_0 + k\tau)\,\delta[t - (t_0 + k\tau)]. \quad (1$$

The first one carries out a linear smoothing. The second one is a sampling, with steps of $\tau = NT_s/(n-1)$, over interval $t_0 \leq t \leq t_0 + NT_s$ (section VI-B, Equation 90).

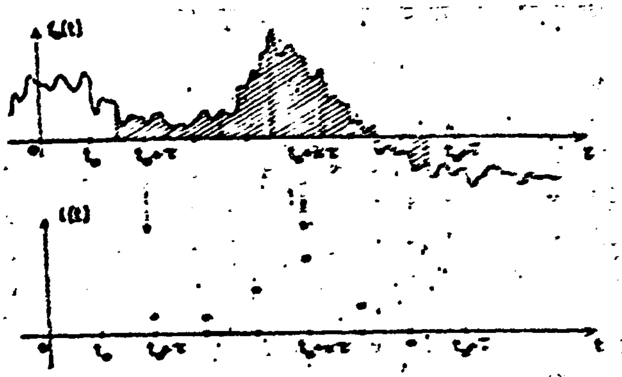

Figure 14. Smoothing principle by using integration of limited range.

A. Properties of Linear Smoothing.

Equation (107) can be written as:

$$l_s(t) = \frac{1}{\tau}\int_{-\infty}^{+\infty} f_s(t-\theta')\, k(\theta')\, d\theta'.$$

with: $k(\theta') = 1$ for $-\frac{\tau}{2} \leq \theta' \leq +\frac{\tau}{2}$

$= 0$ for $|\theta'| > \frac{\tau}{2}$.

We therefore have, by putting $g(t) = (1/\tau)\, k(t)$:

$$l_s(t) = f_s(t) * g(t),$$

and linear smoothing of the gain:

$$G(\nu) = \frac{\sin \pi\nu\tau}{\pi\nu\tau}. \tag{109}$$

a. Effect on the signal.

Since the gain is real, the sinusoidal wave is reproduced without phase shift, but with an amplitude reduced by a ratio of $\sin \pi\nu\tau/\pi\nu\tau$.

b. Effect on the noise.

At every point, the standard deviation σ_1 is given exactly by:

$$\sigma_1^2 = \int_0^{\infty} A(\nu) \left(\frac{\sin \pi\nu\tau}{\pi\nu\tau}\right)^2 d\nu, \tag{110}$$

where $A(\nu)$ is the spectral energy density of the noise at the input of the integrator. Such smoothing by integration makes sense only if $A(\nu)$ is practically constant in the frequency interval where $(\sin \pi\nu\tau/\pi\nu\tau)^2$ is noticeably different from zero. Therefore, to a very good approximation, we have:

$$\sigma_1 \doteqdot \sqrt{\frac{A_0}{2\tau}} \tag{111}$$

where A_o is the spectral energy density of the noise at frequency zer

By the way, we obtain just such a standard deviation when we integrate a DC signal over a time interval τ.

B. Integration Over a Limited Range and Convolution by $g(t) = \frac{2}{n-1}\cos$

First of all, we have to note that the useful observation duration is no longer $T = NT_o$ but a T' between T and $T-\tau$, because of the fact that the two ranges of integration at the extremities of the interval of observation could be truncated, hence, unusable.

(This amounts to an effect analogous to that transient region in the case of a physical filter.) In fact we generally have τ much smaller than T and this correction will be omitted in the following.

The function $l_o(t)$, the result of a linear smoothing of $f_c(t)$ by limited range integration, is then sampled, per Equation (108), and convoluted with:

$$g(t) = \frac{2}{n-1} \cos 2\pi \nu_0 t.$$

The equations of section VI-B are directly applicable by repla ing $f_o(t)$ by $l_o(t)$, and in particular, $A(\nu)$ by:

$$B(\nu) = A(\nu)\, G^2(\nu) \doteqdot A_0 \left(\frac{\sin \pi\nu\tau}{\pi\nu\tau}\right)^2.$$

After integration over the limited range, sampling, and convo tion with a sinusoidal wave, the standard deviation σ at all point is then given by:

$$\sigma^2 = \int_{-\infty}^{+\infty} A(\nu) \left(\frac{\sin \pi\nu\tau}{\pi\nu\tau}\right)^2 \left(\frac{\sin \pi(\nu - \nu_0)(n-1)\tau}{(n-1)\sin \pi(\nu - \nu_0)\tau}\right)^2 d\nu, \tag{112}$$

which we have represented with a hatched area in Figure 15.

When the interval of observation NT_o is much greater than τ, which is generally the case, the width of the hatchmarked lobes is negligible and Equation (112) can be written, with very good approximation, as:

$$\sigma^2 \doteq \frac{A_0}{NT_0} \sum_{k=-\infty}^{+\infty} \left(\frac{\sin \pi(\nu_0 + k/\tau)\tau}{\pi(\nu_0 + k/\tau)\tau}\right)^2,$$

or:

$$\sigma^2 \doteq \frac{A_0}{NT_0} \left(\frac{\sin \pi\nu_0\tau}{\pi\nu_0\tau}\right)^2 \sum_{k=-\infty}^{+\infty} \left(\frac{1}{1 + k/\nu_0\tau}\right)^2 \tag{113}$$

The minimum noise energy that we would have in an interval τ approaching zero, is:

$$\sigma_m^2 \doteq \frac{A_0}{NT_0},$$

and taking into account that the energy of the signal is also reduced by the ratio:

$$\left(\frac{S'}{S}\right)^2 = \left(\frac{\sin \pi\nu_0\tau}{\pi\nu_0\tau}\right)^2,$$

the relative excess of noise as a function of τ is finally given by:

$$\frac{\sigma_e^2}{\sigma_m^2} \doteq \sum_{k=-\infty}^{+\infty} \left(\frac{1}{1 + k/\nu_0\tau}\right)^2. \tag{114}$$

Figure 16 represents the ratio σ_e/σ_m as a function of T_c/τ, the number of integration ranges per period of the sinusoidal wave.

From Figures 13 and 16 we can therefore determine the combined

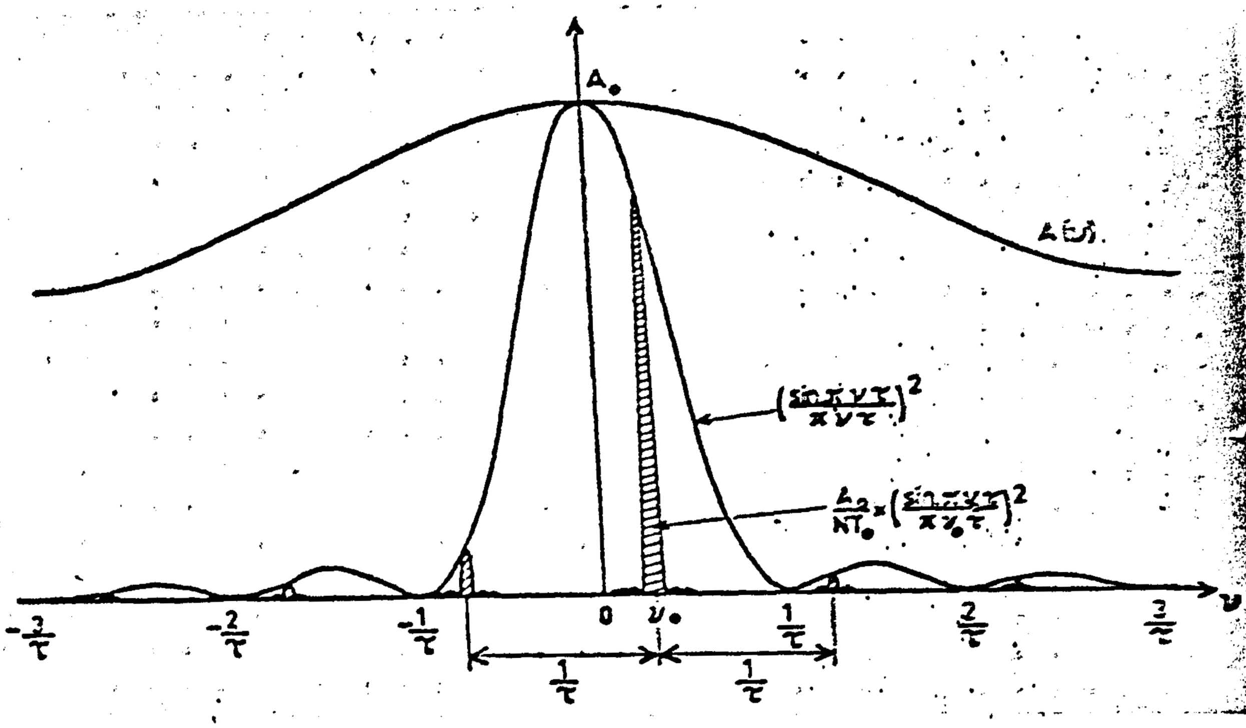

Figure 15. Noise energy after smoothing by integration over limited range followed by convolution with a sinusoidal wave.

effect of the quantization and the sampling inherent in integration over a limited range on the final signal-to-noise ratio. Having plotted on these figures the ordinates corresponding to selected values of σ_o/q and T_o/τ, we form the sum of their squares. As we have seen in section VI-D, the square root of this sum is equal to the ratio σ_T/σ_m, that of the final standard deviation over that which we would have found in the absence of sampling and quantization.

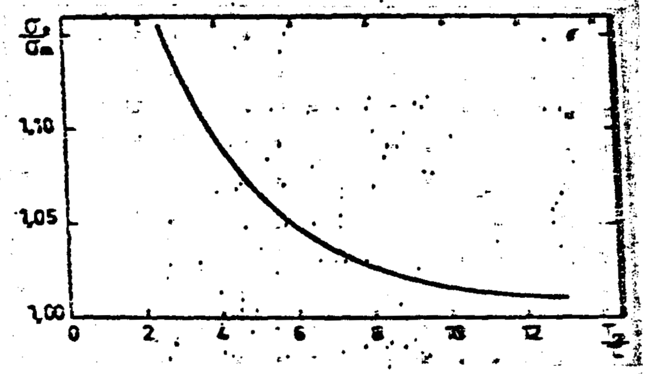

Figure 16. Relative increase in the standard deviation of the noise as a function of the ratio T_o/τ of the period of the signal to the width of the window of integration.

VII. AUTOCORRELATION

This non-linear operation is considered here only for the sake of completeness, because it is much less efficient than convolution by $g(t) = (2/NT_0) \cos 2\pi\nu_0 t$. Furthermore, it involves loss of all information about the phase of this signal. On the other hand, such a process presents the advantage of being able to be applied independently of any initial knowledge of the period $T_0 = 1/\nu_0$ of the signal, and it allows us to evaluate T_0. As such, autocorrelation is useful essentially as a process complementary to convolution, which allows us, due to its initial approximation of T_0, to use the latter.

Let $f_0(t) = s_0(t) + x_0(t)$ be a sum of a sinusoidal signal of frequency ν_0 and of a noise $x_0(t)$. We first of all assume that $f_0(t)$ is observed over an infinite time interval. Its autocorrelation function is by definition:

$$\rho_{f_0}(\tau) = \frac{1}{T \to \infty} \int_{-T}^{+T} f_0(t) \cdot f_0(t+\tau)\, dt = \overline{f_0(t) \cdot f_0(t+\tau)}$$

of:

$$\rho_{f_0}(\tau) = \overline{[s_0(t) + x_0(t)]\,[s_0(t+\tau) + x_0(t+\tau)]}.$$

As the signal and the noise are independent, this average is equal to:

$$\rho_{f_0}(\tau) = \rho_{s_0}(\tau) + \rho_{x_0}(\tau),$$

the sum of the autocorrelation function of the signal and of the noise, with:

$$\rho_{s_0}(\tau) = \overline{S \cos[2\pi\nu_0 t + \varphi] . S \cos[2\pi\nu_0(t+\tau) + \varphi]} = \frac{S^2}{2} \cos 2\pi \nu_0 \tau.$$

Figure 17 shows $\rho_{f_0}(\tau)$ and we can see from it that for a sufficiently large τ (of the order of the inverse of the bandpass of the filter), this function approaches $(S^2/2) \cos 2\pi\nu_0 \tau$. From this we

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deduce ν_0 and S. We could further evaluate $\rho_n(\tau)$ independently either by calculation or from recordings of the noise alone, and to subtract it from $\rho_f(\tau)$ in order to obtain $\rho_s(\tau)$ directly.

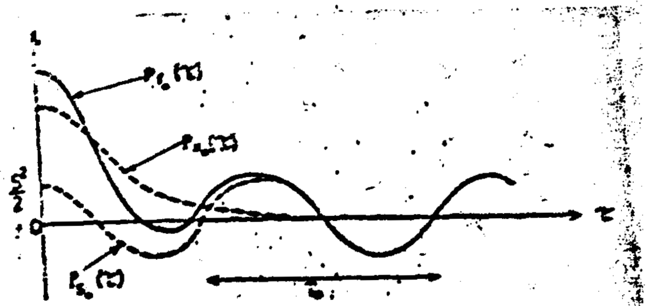

Figure 17. Autocorrelation function of the sum of the sinusoidal wave and of noise.

The interval of observation is in fact limited, and the result is, on the one hand, a distortion of $\rho_f(\tau)$, and, on the other hand, a certain dispersion $\sigma_A^2(\tau)$ of the values of $\rho_f(\tau)$ relative to the average $\rho_f(\tau)$. This dispersion is, in effect, a function of τ, since the interval of observation on which $\rho_f(\tau)$ is evaluated, is itself a function of τ:

$$\rho_f(\tau) = \int_{t_0}^{t_0+T-\tau} f_0(t) \cdot f_0(t+\tau)\, dt.$$

It therefore seems to be difficult to evaluate the statistical properties of S' obtained as a square root of the amplitude of $\rho_f(\tau)$.

Y. W. Lee [16] has studied the particular case where, on the one hand, f(t) is sampled at random intervals sufficiently wide so that the values of the noise should be independent, and, on the other hand, the number of points n sampled for evaluating $\rho_f(\tau)$, is the same regardless of τ. Under these conditions, he has evaluated the standard deviation σ_A of the values of $\rho_f(\tau)$ relative to the average and has compared the ratio S^2/σ_A to the initial signal-to-noise ratio S/σ. The quantity:

$$g_A = \frac{S^2/\sigma_A}{S/\sigma},$$

which expressed the degree of improvement in "visibility" of the signal relative to noise, provides one indication of the gain achieved by autocorrelation. Y. W. Lee has shown that g_A is both a function and of S/σ, and can be less than 1 when the number of points sampled is insufficient, taking S/σ into account. Moreover, compared to t convolution with a sinusoidal wave, carried out under the same random sampling conditions, autocorrelation leads to a loss of "visibility" which is greater as the initial signal-to-noise ratio is smaller. For example, for $S/\sigma = 0.1$, we need 100 times more points in the second case than in the first in order to obtain the same gain of 10.

Finally, we find in [17] the following equation:

$$G = \sqrt{2 + \left(\frac{\sigma}{S}\right)^2},$$

expressing the relative gain of the convolution with respect to autocorrelation in the case where the sampling is periodic, and the other initial conditions remain the same.

APPENDIX

CORRELATION COEFFICIENT BETWEEN TWO VALUES OF $y_{t_0}(t)$ SEPARATED BY τ.

With the notation of the first section, and with reference to Figure 2, we propose to evaluate the correlation coefficient between two values $y_{t_0}(t_0 + \alpha)$ and $y_{t_0}(t_0 + \alpha + \tau)$ of the non-stationary random function:

$$y_{t_0}(t) = [x_0(t) . h(t - t_0)] * g(t). \tag{13}$$

As this correlation coefficient is defined by the normalized quantity:

$$C_\alpha(\tau) = \frac{R_\alpha(\tau)}{R_\alpha(0)} \qquad \text{(A-1}$$

we should evaluate:

$$R_\alpha(\tau) = \overline{y_{t_0}(t_0 + \alpha)\, y_{t_0}(t_0 + \alpha + \tau)}, \qquad \text{(A-2}$$

with the average taken over an ensemble of trials obtained by varyin the central position t_o of the window $h(t-t_o)$ from $-\infty$ to $+\infty$. Fr Equation (13) written as:

$$y_{t_0}(t) = \int_{-\infty}^{+\infty} x_0(t - \theta)\, h(t - \theta - t_0)\, g(\theta)\, d\theta,$$

we have by definition:

$$R_\alpha(\tau) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \overline{x_0(t_0 + \alpha - \theta)\, x_0(t_0 + \alpha + \tau - \theta')} \; h(\alpha - \theta)\, h(\alpha + \tau - \theta')\, g(\theta)\, g(\theta')\, d\theta\, d\theta'. \qquad \text{(A-}$$

Now we perform a change in variables:

$$\theta' = \theta + \varepsilon,$$

which leads to:

$$R_\alpha(\tau) = \int_{-\infty}^{+\infty} \rho(\varepsilon - \tau)\, d\varepsilon \int_{-\infty}^{+\infty} h(\alpha - \theta)\, g(\theta)\, h(\alpha + \tau - \theta - \varepsilon)\, g(\theta + \varepsilon)\, d\theta. \qquad \text{(A}$$

The second integral is the scalar product:

$$I_2 = < h(\alpha - \theta)\, g(\theta),\; h(\alpha + \tau - \theta - \varepsilon)\, g(\theta + \varepsilon) >. \tag{A-5}$$

of two real functions. We can therefore express it in the form of a scalar product of the Fourier transform of one and the conjugate of the Fourier transform of the other [2]:

$$I_2 = < TF\,[h(\alpha - \theta)\, g(\theta)],\; TF^*\,[h(\alpha + \tau - \theta - \varepsilon)\, g(\theta + \varepsilon)] >, \tag{A-6}$$

Taking into account Equation (18), we have:

$$\begin{aligned} TF\,[h(\alpha - \theta)\, g(\theta)] &= [H(-\nu)\, e^{2\pi i \nu \alpha}] * G(\nu) = J_\alpha(\nu) \\ TF^*\,[h(\alpha + \tau - \theta - \varepsilon)\, g(\theta + \varepsilon)] &= e^{2\pi i \nu \varepsilon} \{ [H(-\nu)\, e^{2\pi i \nu(\alpha+\tau)}] * G(\nu) \}^* \\ &= e^{2\pi i \nu \varepsilon} . J^*_{\alpha+\tau}(\nu) \end{aligned} \tag{A-7}$$

which leads to:

$$I_2 = < J_\alpha(\nu),\, e^{2\pi i \nu \varepsilon} . J^*_{\alpha+\tau}(\nu) >,$$

and

$$R_\alpha(\tau) = \int_{-\infty}^{+\infty} \rho(\varepsilon - \tau)\, e^{2\pi i \nu \varepsilon}\, d\varepsilon \int_{-\infty}^{+\infty} J_\alpha(\nu) . J^*_{\alpha+\tau}(\nu)\, d\nu. \tag{A-8}$$

Now, with the change in variables:

$$\varepsilon' = \varepsilon - \tau.$$

Equation A-8 can be written as:

$$R_\alpha(\tau) = \int_{-\infty}^{+\infty} \rho(\varepsilon')\, e^{2\pi i \nu \varepsilon'}\, d\varepsilon' \int_{-\infty}^{+\infty} J_\alpha(\nu) . J^*_{\alpha+\tau}(\nu)\, e^{2\pi i \nu \tau}\, d\nu,$$

with, analogously to Equation (7):

$$\int_{-\infty}^{+\infty} \rho(\varepsilon')\, e^{2\pi i \nu \varepsilon'}\, d\varepsilon' = \frac{1}{2} A(\nu).$$

Finally:

$$\boxed{R_\alpha(\tau) = \frac{1}{2} \int_{-\infty}^{+\infty} A(\nu)\, J_\alpha(\nu)\, J^*_{\alpha+\tau}(\nu)\, e^{2\pi i \nu \tau}\, d\nu} \tag{A-9}$$

We can verify that, by setting $\tau = 0$, we indeed have the Equation (20):

$$R_s(0) = \sigma_s^2 = \int_0^{\infty} A(\nu) |J_s(\nu)|^2 \, d\nu.$$

<u>Application to the Case of Convolution with a Sinusoidal Wave with Limited Recording Duration.</u>

The preceding is applied to the particular case considered in section II: the recording of the noise $x_o(t)$, observed on an interval T, is convoluted with $g(t) = (2/T) \cos 2\pi\nu_o t$. Under these conditions:

$$J_s(\nu) = \frac{\sin \pi(\nu + \nu_o) T}{\pi(\nu + \nu_o) T} e^{\pi i (\nu+\nu_o) T} + \frac{\sin \pi(\nu - \nu_o) T}{\pi(\nu - \nu_o) T} e^{\pi i (\nu-\nu_o) T}. \quad (35)$$

By replacing in Equation A-9 the functions $J_s(\nu)$ and $J^*_{s}(\nu)$ by their values, we obtain:

$$\boxed{\begin{aligned} R_s(\tau) = \frac{1}{2} \int_{-\infty}^{+\infty} A(\nu) \Bigg[& \left(\frac{\sin \pi(\nu + \nu_o) T}{\pi(\nu + \nu_o) T} \right)^2 e^{-2\pi i \nu \tau} + \left(\frac{\sin \pi(\nu - \nu_o) T}{\pi(\nu - \nu_o) T} \right)^2 e^{2\pi i \nu \tau} \\ & + 2 \frac{\sin \pi(\nu + \nu_o) T . \sin \pi(\nu - \nu_o) T}{\pi(\nu + \nu_o) T . \pi(\nu - \nu_o) T} \cos 2\pi\nu_o (2\tau + \tau) \Bigg] d\nu. \end{aligned}} \quad (A-$$

Neglecting the cross term, which is smaller as T is greater relative to $1/\nu_o$, and as we have seen in section II, this equation takes the approximate form:

$$R_s(\tau) \doteq R(\tau) = \cos 2\pi\nu_o \tau \int_{-\infty}^{+\infty} A(\nu) \left[\left(\frac{\sin \pi(\nu + \nu_o) T}{\pi(\nu + \nu_o) T} \right)^2 + \left(\frac{\sin \pi(\nu - \nu_o) T}{\pi(\nu - \nu_o) T} \right)^2 \right] d\nu \quad (A-$$

from which we get the value of the correlation coefficient:

$$\boxed{C_s(\tau) \doteq C(\tau) = \frac{R(\tau)}{R(0)} = \cos 2\pi\nu_o \tau.} \quad (A$$

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